

The Formal System $\lambda\delta$

FERRUCCIO GUIDI

Department of Computer Science, University of Bologna, Italy

The formal system $\lambda\delta$ is a typed λ -calculus that pursues the unification of terms, types, environments and contexts as the main goal. $\lambda\delta$ takes some features from the Automath-related λ -calculi and some from the pure type systems, but differs from both in that it does not include the Π construction while it provides for an abbreviation mechanism at the level of terms. $\lambda\delta$ enjoys some important desirable properties such as the confluence of reduction, the correctness of types, the uniqueness of types up to conversion, the subject reduction of the type assignment, the strong normalization of the typed terms and, as a corollary, the decidability of type inference problem.

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Additional Key Words and Phrases: Abbreviations, terms as types, environments as terms

To Silvia,
a very special lady

1. INTRODUCTION

The leading goal at the root of the present work is the design of a typed λ -calculus, to be used as a logical framework, featuring the unification of terms, types and environments (with the terminology of [Sørensen and Urzyczyn 2006]) while enjoying a desirable meta-theory in the sense of [Barendregt 1993]. In principle we pursue this unification, whose benefits we discuss in Subsection 1.1, by defining a suitable set of expressions that can be terms, types and environments at the same time.

The purpose of this paper is to report on our first attempt to realize such a calculus. In Subsection 1.2 we summarize our starting points and our achievements.

In Subsection 1.3 we briefly introduce the digital specification of our calculus and of its theory inside the Calculus of Inductive Constructions (CIC) [Guidi 2007a]. This specification has been checked by two CIC-based proof assistants.

The calculus is defined in Section 2 where the syntax, the reduction rules and the type assignment rules are given. Our main theorems on the calculus are presented in Section 3. In Section 4 we extend our calculus by adding an “exclusion” binder, which we show an application of. The concluding remarks are in Section 5.

This paper includes four appendices: in Appendix A we show an application of our calculus as a theory of expressions for the structural fragment of the Minimal

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Type Theory [Maietti and Sambin 2005], while in Appendix B the author proposes to push the calculus in the direction of the “environments as terms as types” paradigm until the unification of these three concepts is reached.

In Appendix C we report on the differences between the version of the calculus in front of the reader and its initial version [Guidi 2006].

In Appendix D we give the pointers to the digital version of our results.

1.1 Background and Motivations

Untyped λ -calculus [Church 1941] was introduced by Church as a theory of computable functions. Adding a very simple type theory to this calculus, where types are never created by abstraction, Curry obtained a version of the simply typed λ -calculus $\lambda \rightarrow$ (a different version of $\lambda \rightarrow$ was proposed by Church afterwards).

Typing by abstraction was introduced in the second half of the past century in response to the need of improving the expressiveness of the above type theory, and this gave rise to many λ -calculi typed more powerfully. The type of a term is always assigned in an environment, that is a structure holding the type information on the free variables that may occur in that term [Sørensen and Urzyczyn 2006].

An historical survey on type theory can be found in [Kamareddine et al. 2004].

In some theories a type can be treated as a term and can be given a type, which is usually termed a kind. Nevertheless many calculi, especially those of the Pure Type Systems (PTS) tradition [Barendregt 1993], provide for constructions that build types, or kinds, but not terms. This is the case of the so-called Π construction. Moreover terms and environments usually belong to distinct syntactical categories.

One reason for having different constructions for terms and types lays in the so-called “Propositions As Types and Proof As Terms” (PAT) interpretation [Kamareddine et al. 2004] (also known as the Curry-Howard isomorphism) and in the general consensus that propositions and proofs have a significantly different structure. We recall that according to the PAT interpretation, a typed λ -calculus can serve as a logical framework where a proposition is encoded in a type whose inhabitants encode the proofs of that proposition.

On the other hand there are scenarios in which one wants to encode a proposition in a term or a proof in a type. We call this situation: the reverse PAT interpretation.

—The Automath experience.

Historically the embedding of logic inside λ -calculus does not always follow the PAT interpretation. This is the case of Aut – 68 [van Benthem Jutting 1994b]: a language of the Automath family [de Bruijn 1994c] that is very close to a λ -calculus. This language has only one kind, named `type`, and this forces the embedding of logic clearly shown in [de Bruijn 1994a], which is used throughout the formal specification of Landau’s Grundlagen [van Benthem Jutting 1994a].

We summarize the situation in Figure 1. In Aut – 68 the proofs of a proposition do not inhabit the proposition directly, as in the PAT interpretation, but they inhabit the “assertion type” associated to the proposition. In this way a proposition differs from the type of its proofs.

—The realizability tradition.

One of the basic ideas behind type theory is that terms encode some entities (for instance computable functions, computer programs, propositions, proofs) and

Encoding	PAT	Aut – 68	PML
kinds	sort of propositions	type	
types	propositions	sort of propositions, assertion types	realizers
terms	proofs	propositions, proofs	specifications

Fig. 1. Different embeddings of logic in type theory

these entities satisfy a desired property if the corresponding terms are typable. In this respect there are type systems set up to capture some properties of propositions. For instance in the computer program verification scenario one can state that a proposition is admissible if it the specification of a program (this idea is taken from the realizability tradition [Kleene 1945], where the admissible formulae are those having a realizer, i.e. an implementation). In this perspective one may want to encode the propositions in the terms and their realizers or implementations in the types. This is the case of PML [Raffalli 2007a; 2007b; 2008]: an experimental programming language with program verification support. Notice that in PML the standard PAT interpretation is also allowed.

The above considerations lead to think that a type theory intended as a logical framework is more flexible if it supports both PAT interpretations at the same time instead of supporting just one of them (either the standard one or the inverse one).

This result is achieved by designing the type theory in such a way that both terms and types are capable of encoding either a proof or a proposition.

The simplest way to obtain this feature is by allowing on one hand the term constructions at the level of types and on the other hand the type constructions at the level of terms. By so doing, we are naturally led to unify terms and types.

It is worth remarking that this unification already appears to some extent in a number of works including [de Bruijn 1994c; Nederpelt 1994; de Vrijer 1994; van Benthem Jutting 1994c; Coquand 1985; Kamareddine 2005].

Coming now to the treatment of environments, there are well established motivations for allowing these structures to contain not just declarations, but abbreviations (i.e. non-recursive definitions) as well. We mention the following ones.

— **Practically unavoidable.**

Abbreviations allow to factorize large terms increasing their readability. It is a matter of fact that Mathematics is unimaginable without abbreviations and for this reason every type theory designed as a realistic foundation for developing Mathematics includes some kind of abbreviation mechanism. Taking three very different examples of such theories, we can mention the Automath languages [de Bruijn 1994c], Constructive Type Theory [Nordström et al. 1990] and the Calculus of Inductive Constructions [Coquand and Paulin-Mohring 1990].

— **Efficient reduction.**

Abbreviations allow to write the β -contraction in the call-by-name style [Curien and Herbelin 2000] “ $(\lambda x:W.t)(v) \rightarrow_{\beta} \text{let } x = v \text{ in } t$ ” with the effect of delaying the substitution of v in t . This feature is a crucial ingredient of optimal reduction strategies [Asperti and Guerrini 1999] and is exploited in real reduction machines.

Very convenient extensions of well established calculi by means of abbreviations are presented in [Kamareddine et al. 1999; Curien and Herbelin 2000].

Once environments are equipped with abbreviations, we see motivations for pursuing a full duality between environments and terms.

- **Aggregates without inductive types.** Aggregate data structures, or aggregates for short, play a central role both in programming languages (where they appear as records, modules or objects) and in Mathematics (where they appear as mathematical structures). The type theories featuring aggregates as terms usually exploit inductive types for this purpose, but the machinery for supporting inductive types is too complex if one is only interested in supporting aggregates [de Bruijn 1991], especially if dependent types are allowed. On the other hand every type theory has some support for environments and an environment with abbreviations can serve as an aggregate with dependent components. In this respect we conjecture that supporting environments as terms is much simpler than supporting inductive types for the only purpose of having aggregates as terms.
- **The $\lambda\mu$ tradition.** Beside terms, types and environments, the λ -calculi for the PAT interpretation of classical logic derived from $\lambda\mu$ [Sørensen and Urzyczyn 2006] include structures called “contexts” that play the role of continuations in functional programming. The most general of these calculi, $\bar{\lambda}\mu\tilde{\mu}$ [Curien and Herbelin 2000], features abbreviations in contexts (but not in terms) and a duality between terms and contexts, which yet does not yield the unification of the two. On the other hand we conjecture that contexts can be easily injected into environments with abbreviations if these environments are also equipped with other constructions usually found in terms (for instance applications). Such extended environments become very close to terms themselves and may be realized by pursuing a “terms as environments” discipline in the design of the type theory.

1.2 Outline

This paper describes a typed λ -calculus, that we call $\lambda\delta$ after the names of its binders, that aims at the unification of terms, types, kinds and environments both in a static sense and in a dynamic sense. The static unification lays on the use of a suitable set of expressions that can represent terms, types, kinds and environments simultaneously. Additionally, the dynamic unification lays on allowing the same reduction steps on these expressions whatever they represent.

We are interested in respecting the following desirable constraints: this calculus must have a well conceived meta-theory, which includes the commonly required properties and, as a logical framework, must have enough flexibility and expressive power to encode a non-trivial fragment of Mathematics in a realistic manner.

The above considerations imply that the design of $\lambda\delta$ involves two crucial aspects: the choice of the expressions and the choice of the reduction steps allowed on the expressions. In this section we want to discuss these aspects and to analyze their impact on the capability of our calculus to meet the requirements we have set.

The set of the expressions. Our approach in this paper is to build expressions using a reasonably small set of constructions, which we plan to extend in the future.

The starting point is the calculus Λ_∞ [van Benthem Jutting 1994c] where a set Δ of expressions is generated by a sort τ , variable occurrences, binary applications and typed λ -abstractions in which the types themselves are expressions in Δ .

This is a very basic platform to which we apply the following modifications.

Firstly we add untyped abbreviations, like “let $x = v$ in t ”, following the motivation outlined in Subsection 1.1. Secondly we notice that the presence of untyped sorts (as τ in Λ_∞ or as \square in the λ -Cube [Barendregt 1993]) complicates the meta-theory unnecessarily because a distinction must be made between the legal expressions having a type and the legal expressions not having a type. To overcome this drawback we use an infinite number of sorts in place of the single sort τ and we set up a type system (see below) in which every sort is typed. Thirdly we add explicit type annotations (also known as “explicit type casts” in some programming languages) to obtain another meta-theoretical benefit: with these constructions we easily reduce the type checking problem to the type inference problem.

The main limitation of the above set of constructions is the absence of the higher-order abstraction (i.e. the Π construction of the shapes $(\square, *)$ and (\square, \square) according to Barendregt’s classification), which essentially sets the expressive power of $\lambda\delta$ to that of λP [Barendregt 1993].¹ In any case we can assume that this power is enough to encode non-trivial parts of Mathematics [van Benthem Jutting 1994a].²

We also set the additional limitation that a variable occurrence is not an environment constructor because the interpretation of an expression like “ $\lambda x:W.x$ ” as an environment is not straight-forward at all (here W stands for an expression). However in Appendix B.1 we give some hints on how we plan to face this problem.

As a consequence we use two sets of expressions, one for the terms (that also serve as types and kinds) and one for the environments, which is a proper subset of the former. This means that $\lambda\delta$ realizes the unification of types and terms, which is the focus of the calculus, but it does not realize the unification of environments and terms yet. Namely environments are just expressions formally generated by some term constructors, but $\lambda\delta$ has no support for using them as terms.

It is important to notice that $\lambda\delta$ differs from the Automath-related λ -calculi [Nederpelt et al. 1994] in that they do not provide for an abbreviation construction at the level of terms. We also notice that when abbreviations are used, the λ -abstraction it is not strictly necessary for building a logical framework. This is the case of PAL^+ [Luo 2003]: a platform where partial applications of functions are not allowed. As a matter of fact, partial applications have well established benefits in several contexts including practical functional programming, so our choice is definitely to include the λ -abstraction in our calculus.

The set of the reduction schemes. The reduction schemes aim at realizing deterministic and confluent computations (as the ones of Λ_∞), so critical pairs are avoided for simplicity. Since $\lambda\delta$ is not focused on achieving the unification of terms and environments, its reduction schemes work only on terms and no support is given for the reduction of environment constructors. Nevertheless these schemes are designed following the principle that they should also work on environments when possible. In particular we must be aware that an environment is essentially a list of declarations (that we represent with λ -abstractions) and abbreviations whose position must be preserved when the environment is reduced.

For this reason we use the call-by-name β -contraction scheme in place of its

¹Currently we do not have a proof of this statement, but our conjecture is based on the general consensus that Λ_∞ has the expressive power of λP [Barendregt 1993].

²We are aware that Aut – QE is a bit more powerful than λP [Kamareddine et al. 2004].

call-by-value version (the one used by Λ_∞) because the λ -abstraction in the reduct becomes an abbreviation in the reductum instead of being deleted. Another advantage of the call-by-name β -reduction is discussed in Subsection 1.1.

Moreover we have three reduction schemes working on abbreviations: namely a δ -expansion to unfold an abbreviation without removing it, a ζ -contraction for removing an unreferenced abbreviation (this reduction would not be allowed if the abbreviation were an environment constructor) and a ν -swap for permuting an application-abbreviation pair as in [Curien and Herbelin 2000].

Finally we have a τ -contraction for removing explicit type annotations.

Remarkably we do not consider the η -contraction. This is a choice of many calculi including Λ_∞ and the systems of the λ -Cube [Barendregt 1993].

Also notice that we can obtain a call-by-value β -contraction by concatenating a call-by-name β contraction, a δ -expansion and a ζ -contraction.

The type system. Our aim is to confine the dynamic aspect of the type assignment in the so-called “conversion rule” [Barendregt 1993]. This means that we wish to remove any reference to reduction from the other type assignment rules. The technical benefit of this approach is that we make clear syntactical distinction between the construction steps and the conversion steps needed to infer a type.

Typed sorts. We have a sequence of sorts $h \mapsto \text{Sort}_h$ (where h ranges over the set \mathbb{N} of the natural numbers) and a function $g : \mathbb{N} \rightarrow \mathbb{N}$ that we can choose at will as long as $h < g(h)$ holds for every h . In this setting Sort_h is typed by $\text{Sort}_{g(h)}$.

Typed variable occurrences. We exploit the idea that an unreferenced variable needs a legal declaration only if it is the formal argument of a function, to combine the so-called “start rule” and “weakening rule” [Barendregt 1993] in a simpler rule.

Typed λ -abstractions. We use the policy of Λ_∞ , which is known as λ -typing. Namely up to conversion, the type of a λ -abstraction is a λ -abstraction. This policy is adopted by many calculi of the Automath family [Nederpelt et al. 1994] and by other calculi including [Kamareddine 2005; de Groote 1993; Wiedijk 1999].

Typed abbreviations. We use the λ -typing pattern with abbreviations in place of λ -abstractions. This approach yields a uniform typing policy for both binders.

Typed applications. We use the “compatible” application rule of [Kamareddine et al. 1999] with λ in place of Π , because it does not involve reduction. By so doing, we strengthen the so-called “applicability condition”³ with respect to Λ_∞ , but we conjecture that this is a minor drawback. For instance the term $t \equiv (x_1 z)$ is legal in the environment $\Gamma \equiv (x_0 : \lambda y : \tau. y), (x_1 : x_0), (z : \tau)$ for Λ_∞ but not for $\lambda\delta$.

Explicit type annotations. We use a “compatible typing” policy as well.

The meta-theoretical properties. One of the aims of the present paper is to show that the design features of $\lambda\delta$ we just described are compatible with the presence of a desirable meta-theory in the usual sense. The main results are:

- the reduction is confluent (Church-Rosser property): Theorem 3(3);
- the reduction is safe (subject reduction property): Theorem 9(2);
- the typed terms are strongly normalizing: Theorem 10(2).

³This is the condition that an application must satisfy in order to be legal or well typed.

We also prove other standard properties like the correctness of types, the uniqueness of types up to reduction and the decidability of type the inference problem.

We stress that the λ -abstraction is predicative in that $\Gamma \vdash \lambda x:W.t : W$ never holds. So $\lambda\delta$ can serve as a theory of expressions for the type theories requiring a meta-language with a predicative abstraction like those in the Marin-Löf style [Maietti and Sambin 2005; Nordström et al. 1990; Martin-Löf 1984]

1.3 The Certified Specification

The initial version of $\lambda\delta$ appears in [Guidi 2006] where the author outlines the definitions used in [Guidi 2007a] to specify an extension of $\lambda\delta$ named $\chi\lambda\delta$ (see Section 4) in the Calculus of Inductive Constructions (CIC). Using this encoding it is possible to certify all currently proved properties of $\chi\lambda\delta$ with the CIC-based proof assistants COQ [Coq development team 2007] and MATITA [Asperti et al. 2006].

Following the description of Λ_∞ in [van Benthem Jutting 1994c], the CIC specification exploits position indexes [de Bruijn 1994b] rather names to represent the bound variable occurrences. However in this paper we will use names.

Remarkably $\chi\lambda\delta$ was born and developed in the digital format of [Guidi 2007a], which is not the formal counterpart of some informal material previously written on paper (as it happens for most of currently digitalized Mathematics). In particular the detailed proofs of the properties of $\chi\lambda\delta$ currently exist only in their digital version. Producing a hard copy of these proofs is indeed an interesting challenge because it requires the implementation of a suitable technology for the mechanical transformation of digital CIC proof terms into human-readable proofs written in L^AT_EX format.⁴ Our estimation on the length of the hard copy is: 600 pages.

In this paper we outline all proofs of our statements by reporting on the proof strategy and on the main dependences of each proof. Most proofs are by induction on the length of a derivation or by cases on the last step of a derivation. Very often both techniques are applied together. This procedure breaks the proof in lot of cases which we do not give the details of (because they are very easy). However we report on the interesting cases giving some hints on how they are solved.

In Appendix D we give the pointers to the digital proof objects representing the proofs mentioned in the paper. These proof objects are available as resources of the Hypertextual Electronic Library of Mathematics (HELM) [Asperti et al. 2003].

In Appendix C we present the main advancements of [Guidi 2007a] at its current state over the description given in [Guidi 2006].

2. THE DESCRIPTION OF $\lambda\delta$

In this section we will define $\lambda\delta$ in terms of its grammar (Subsection 2.1), its reduction rules (Subsection 2.3) and its native type assignment rules (Subsection 2.4). We will also define some relevant auxiliary notions such as the static type assignment (Subsection 2.5), the arity assignment (Subsection 2.6) and two preorders on environments (Subsection 2.7). Care was taken to order these topics in a way that takes the reader to the native type assignment rules as soon as possible.

⁴In [Guidi 2007b] we present an effective procedure for transforming a CIC proof term in a sequence of basic proof steps. We already implemented this procedure in the proof assistant MATITA.

$\lambda\delta$ uses three data types: the set \mathbb{N} of the *natural numbers*, the set \mathbb{T} of the *terms* and the set \mathbb{E} of the *environments*. \mathbb{N} is used to represent sort indexes (all indexes start at 0), \mathbb{T} contains the expressions the calculus is about (also called pseudo-terms) and \mathbb{E} can be seen as a subclass of \mathbb{T} . Although it is not strictly necessary, it is convenient to present \mathbb{T} and \mathbb{E} as two distinct data types.

In the presentation of $\lambda\delta$ in front of the reader, the term variables are referenced by name and the names for these variables (i.e. x, y, \dots) belong to a data type \mathbb{V} .

Consistently throughout the presentation, we will be using the following convention about the names of the meta-variables: i, j, h, k will range over \mathbb{N} ; T, U, V, W will range over \mathbb{T} and C, D, E, F will range over \mathbb{E} or will denote a part of an environment. We use the Latin capital letters for the term meta-variables following the untyped λ -calculus tradition [Barendregt 1993] and we use these letters also for the environment meta-variables, instead of using the standard Greek capital letters, because we follow the “environments as terms” policy pursued by $\lambda\delta$.

Lists will also be used (we need them in Subsection 3.2 to prove the strong normalization theorem). The names of variables denoting lists will be overlined: like \overline{T} for a list of terms. We will use \circ for the empty list and the infix semicolon for concatenation: like $T ; \overline{T}$.

In order to avoid the explicit treatment of α -conversion, we will assume that the names of the bound variables and of the free variables are disjoint in every term, judgement and rule of the calculus (this is known as the “Barendregt convention”).

2.1 The Language

Our syntax of terms and environments takes advantage of the so-called *item notation* [Kamareddine and Nederpelt 1996b] because of its well documented benefits. When using the item notation of λ -terms, the operands of an application are presented in reverse order with respect to standard notation, i.e. the application of T to V is presented like $(T V)$ in standard notation and like $(V).T$ in item notation. This means that a β -redex takes the form $(V).\lambda x:W.T$ rather than $(\lambda x:W.T V)$. In this situation the argument V and the abstraction $\lambda x:W$ are close to each other rather than having the body T between them, which can be very long. In this sense we believe that this notation, which is almost a constant of the Automath-related works [Nederpelt et al. 1994], improves the visual understanding of β -redexes by helping the reader to find the argument-abstraction pairs more easily.

DEFINITION 1 TERMS AND ENVIRONMENTS.

The terms of $\lambda\delta$ are made of these syntactical items: Sort_h (sort), x (variable occurrence), $\lambda x:W$ (abstractor), $\delta x \leftarrow V$ (abbreviator), (V) (applicator) and $\langle W \rangle$ (type annotator). The sets of terms and environments are defined as follows:

$$\mathbb{T} \equiv \text{Sort}_{\mathbb{N}} \mid \mathbb{V} \mid \lambda \mathbb{V}:\mathbb{T}.\mathbb{T} \mid \delta \mathbb{V} \leftarrow \mathbb{T}.\mathbb{T} \mid (\mathbb{T}).\mathbb{T} \mid \langle \mathbb{T} \rangle.\mathbb{T} \quad (1)$$

$$\mathbb{E} \equiv \text{Sort}_{\mathbb{N}} \mid \lambda \mathbb{V}:\mathbb{T}.\mathbb{E} \mid \delta \mathbb{V} \leftarrow \mathbb{T}.\mathbb{E} \mid (\mathbb{T}).\mathbb{E} \mid \langle \mathbb{T} \rangle.\mathbb{E} \quad (2)$$

In the above definition Sort_h is the sort of index h , x is a variable occurrence, $\lambda x:W.T$ is the usual λ -abstraction (simply abstraction henceforth) of T over the type W , $\delta x \leftarrow V.T$ is the abbreviation of V in T (i.e. let $x = V$ in T), $(V).T$ is the application of T to V (i.e. $(T V)$ in standard notation) and $\langle W \rangle.T$ is the type annotation of T with W (i.e. $(T : W)$ in ML notation).

We remark that type annotations allow to reduce the type checking problem to the type inference problem: see Theorem 7(6) and Theorem 8(8).

We can generalize the application to $(V_1 ; \dots ; V_i).T$ that denotes $(V_1) \dots (V_i).T$.

It follows from Definition 1(2) that an environment E is always of the form $C.\text{Sort}_h$, so we allow the notations $E.\lambda x:W$ and $E.\delta x \leftarrow V$ by which we mean the environments $C.\lambda x:W.\text{Sort}_h$ and $C.\delta x \leftarrow V.\text{Sort}_h$ respectively.

A focalized term is an ordered pair (E, T) representing a term T closed in an environment E . In the “environments as terms” perspective pursued by $\lambda\delta$, we can also think that such a pair denotes the concatenation of T after E . Namely $(C.\text{Sort}_h, T)$ may denote the term $C.T$. We stress that focalized terms play an essential role in the substitution lemma for typing, Theorem 8(4), and in the proof that the type inference problem is decidable, Theorem 11(2).

2.2 Some Helper Operators

Now we can introduce some operators that we will use in the next sections.

DEFINITION 2 FREE VARIABLES.

The subset $\text{FV}(T)$ contains the free variables occurring in the term T .

The free variables of a term are defined as usual.

DEFINITION 3 ENVIRONMENT PREDICATE.

The predicate $\text{env}(T)$ states that the term T has the shape of an environment.

—(sort) $\text{env}(\text{Sort}_h)$;

—(compatibility) if $\text{env}(T)$ then

$\text{env}(\lambda x:W.T)$ and $\text{env}(\delta x \leftarrow V.T)$ and $\text{env}(\langle V \rangle.T)$ and $\text{env}(\langle W \rangle.T)$.

We need this predicate only because in $\lambda\delta$ some terms are not environments (see Subsection 1.2) and we use it just in Theorem 12(2).

The substitution operators we define below are exploited by the current reduction rules (see Subsection 2.3), but we conjecture that these rules can be reformulated without mentioning substitution explicitly.

DEFINITION 4 STRICT SUBSTITUTION ON TERMS.

The non-deterministic partial function $[y^+ \leftarrow W]_t T$ substitutes W for one or more occurrences of y in T while it remains undefined if $y \in \text{FV}(W)$ or if $y \notin \text{FV}(T)$.

The subscript “ t ” is part of the notation and the “ $+$ ” recalls “one or more”.

- (1) (var) if $y \notin \text{FV}(W)$ then $[y^+ \leftarrow W]_t y = W$;
- (2) (compatibility) if $[y^+ \leftarrow W]_t V_1 = V_2$ and $[y^+ \leftarrow W]_t T_1 = T_2$ then
 - (a) (abst) $[y^+ \leftarrow W]_t \lambda x:V_1.T = \lambda x:V_2.T$ and $[y^+ \leftarrow W]_t \lambda x:V.T_1 = \lambda x:V.T_2$ and $[y^+ \leftarrow W]_t \lambda x:V_1.T_1 = \lambda x:V_2.T_2$;
 - (b) (abbr) $[y^+ \leftarrow W]_t \delta x \leftarrow V_1.T = \delta x \leftarrow V_2.T$ and $[y^+ \leftarrow W]_t \delta x \leftarrow V.T_1 = \delta x \leftarrow V.T_2$ and $[y^+ \leftarrow W]_t \delta x \leftarrow V_1.T_1 = \delta x \leftarrow V_2.T_2$;
 - (c) (appl) $[y^+ \leftarrow W]_t (V_1).T = (V_2).T$ and $[y^+ \leftarrow W]_t \langle V \rangle.T_1 = \langle V \rangle.T_2$ and $[y^+ \leftarrow W]_t \langle V_1 \rangle.T_1 = \langle V_2 \rangle.T_2$;
 - (d) (cast) $[y^+ \leftarrow W]_t \langle V_1 \rangle.T = \langle V_2 \rangle.T$ and $[y^+ \leftarrow W]_t \langle V \rangle.T_1 = \langle V \rangle.T_2$ and $[y^+ \leftarrow W]_t \langle V_1 \rangle.T_1 = \langle V_2 \rangle.T_2$.

As already pointed out in [Guidi 2006], the function that substitutes W for y in T can be defined in many different ways. The difference lays in the number of occurrences of y that a single application of the function can substitute. The choices are: one, one or more, zero or more, all, all if one exists. Our approach is to adopt the second choice and we can justify it with some technical reasons connected to reduction (see Subsection 2.3). $\lambda\delta$ currently defines two δ -reduction rules (i.e. expansions of local definitions) and we want to use the same substitution function in the description of both rules. This consideration rules out the first choice of the above list because it invalidates Theorem 3(1), that is a prerequisite of Theorem 3(3). The third and the fourth choices, that are the most used in the literature, do not have this problem, but complicate one of the δ -reduction rules if we want to preserve its “orthogonality” (i.e. absence of critical pairs) with respect to the ζ -reduction rule. It is important to stress that this “orthogonality” simplifies the proof of Theorem 3(2): another prerequisite of Theorem 3(3). The last choice of the above list is simply too complex with respect to the benefits it gives.

Notice that with our substitution function we can not replace a variable with itself but this is not a problem since we use this function just to evaluate the δ -redexes, i.e. we use it just to expand non-recursive definitions.

Using the same approach, we can define the strict substitution on environments.

DEFINITION 5 STRICT SUBSTITUTION ON ENVIRONMENTS.

The non-deterministic partial function $[y^+ \leftarrow W]_e E$ substitutes the term W in the environment E for one or more occurrences of the variable y occurring in E .

The subscript “ e ” is part of the notation and the “+” recalls “one or more”.

The rules are the following: if $[y^+ \leftarrow W]_t V_1 = V_2$ and $[y^+ \leftarrow W]_e E_1 = E_2$ then

- (1) (*abst*) $[y^+ \leftarrow W]_e \lambda x:V_1.E = \lambda x:V_2.E$ and $[y^+ \leftarrow W]_e \lambda x:V.E_1 = \lambda x:V.E_2$ and $[y^+ \leftarrow W]_e \lambda x:V_1.E_1 = \lambda x:V_2.E_2$;
- (2) (*abbr*) $[y^+ \leftarrow W]_e \delta x \leftarrow V_1.E = \delta x \leftarrow V_2.E$ and $[y^+ \leftarrow W]_e \delta x \leftarrow V.E_1 = \delta x \leftarrow V.E_2$ and $[y^+ \leftarrow W]_e \delta x \leftarrow V_1.E_1 = \delta x \leftarrow V_2.E_2$;
- (3) (*appl*) $[y^+ \leftarrow W]_e (V_1).E = (V_2).E$ and $[y^+ \leftarrow W]_e (V).E_1 = (V).E_2$ and $[y^+ \leftarrow W]_e (V_1).E_1 = (V_2).E_2$;
- (4) (*cast*) $[y^+ \leftarrow W]_e \langle V_1 \rangle.E = \langle V_2 \rangle.E$ and $[y^+ \leftarrow W]_e \langle V \rangle.E_1 = \langle V \rangle.E_2$ and $[y^+ \leftarrow W]_e \langle V_1 \rangle.E_1 = \langle V_2 \rangle.E_2$.

The strict substitution on focalized terms is defined following the same pattern.

DEFINITION 6 STRICT SUBSTITUTION ON FOCALIZED TERMS.

The non-deterministic partial function $[y^+ \leftarrow W]_f (E, T)$ substitutes W in (E, T) for one or more occurrences of the variable y occurring in (E, T) .

The subscript “ f ” is part of the notation and the “+” recalls “one or more”.

The rules are the following: if $[y^+ \leftarrow W]_e E_1 = E_2$ and $[y^+ \leftarrow W]_t T_1 = T_2$ then $[y^+ \leftarrow W]_f (E_1, T) = (E_2, T)$ and $[y^+ \leftarrow W]_f (E, T_1) = (E, T_2)$ and $[y^+ \leftarrow W]_f (E_1, T_1) = (E_2, T_2)$.

The strict substitution on focalized terms is needed to state the substitution lemma for the native type assignment in a way that breaks the mutual dependences existing between the analogous lemmas stated just for the strict substitution on terms and on environments (see Theorem 8).

$$\begin{array}{c}
\frac{}{T \Rightarrow T} \text{ refl} \quad \frac{W_1 \Rightarrow W_2 \quad T_1 \Rightarrow T_2}{\lambda x:W_1.T_1 \Rightarrow \lambda x:W_2.T_2} \text{ abst} \quad \frac{V_1 \Rightarrow V_2 \quad T_1 \Rightarrow T_2}{\delta x \leftarrow V_1.T_1 \Rightarrow \delta x \leftarrow V_2.T_2} \text{ abbr} \\
\frac{V_1 \Rightarrow V_2 \quad T_1 \Rightarrow T_2}{(V_1).T_1 \Rightarrow (V_2).T_2} \text{ appl} \quad \frac{W_1 \Rightarrow W_2 \quad T_1 \Rightarrow T_2}{\langle W_1 \rangle.T_1 \Rightarrow \langle W_2 \rangle.T_2} \text{ cast} \\
\frac{V_1 \Rightarrow V_2 \quad T_1 \Rightarrow T_2}{(V_1).\lambda x:W.T_1 \Rightarrow \delta x \leftarrow V_2.T_2} \beta \quad \frac{V_1 \Rightarrow V_2 \quad T_1 \Rightarrow T_2 \quad [x^+ \leftarrow V_2]_t T_2 = T}{\delta x \leftarrow V_1.T_1 \Rightarrow \delta x \leftarrow V_2.T} \delta \\
\frac{T_1 \Rightarrow T_2 \quad x \notin \text{FV}(T_1)}{\delta x \leftarrow V.T_1 \Rightarrow T_2} \zeta \quad \frac{T_1 \Rightarrow T_2}{\langle W \rangle.T_1 \Rightarrow T_2} \tau \quad \frac{V_1 \Rightarrow V_3 \quad V_2 \Rightarrow V_4 \quad T_1 \Rightarrow T_2}{(V_1).\delta x \leftarrow V_2.T_1 \Rightarrow \delta x \leftarrow V_4.(V_3).T_2} v
\end{array}$$

Fig. 2. Environment-free parallel reduction rules on terms

scheme	redex	reductum
β -contraction	$(V).\lambda x:W.T$	$\rightarrow_\beta \delta x \leftarrow V.T$
δ -expansion	$\delta x \leftarrow V.T$	$\rightarrow_\delta \delta x \leftarrow V.[x^+ \leftarrow V]_t T$ if $x \in \text{FV}(T)$
ζ -contraction	$\delta x \leftarrow V.T$	$\rightarrow_\zeta T$ if $x \notin \text{FV}(T)$
τ -contraction	$\langle W \rangle.T$	$\rightarrow_\tau T$
v -swap	$(V_1).\delta x \leftarrow V_2.T$	$\rightarrow_v \delta x \leftarrow V_2.(V_1).T$

Fig. 3. Environment-free reduction steps

2.3 Reduction and Conversion

The equivalence of terms in $\lambda\delta$ is based on *environment-dependent conversion*, that is the reflexive, symmetric and transitive closure of *environment-dependent reduction*. The latter is expressed in terms of *environment-free reduction*, that is the compatible closure of five reduction schemes named: β , δ , ζ , τ , v .

The purpose of the present section is to describe this construction in detail.

The need for environment-dependent reduction and conversion derives from the presence of abbreviations in environments [Kamareddine et al. 1999]: for example in the environment $E.\delta x \leftarrow V$ we want to δ -expand the term x to V .

DEFINITION 7 ENVIRONMENT-FREE REDUCTION ON TERMS.

The relation $T_1 \Rightarrow T_2$ indicates one step of environment-free parallel reduction from T_1 to T_2 . Its rules are in Figure 2. The reduction steps are in Figure 3.

Environment-free reduction is presented in its parallel form to ease the proof of the Church-Rosser property stated by Theorem 3(2). In fact using parallel reduction, we bypass the necessity to trace redexes as done in [Barendregt 1993].

The effect of a step $T_1 \Rightarrow T_2$ is to reduce a subset of the redexes appearing in T_1 .

The β scheme does not perform a full β -contraction in the usual sense, but converts a β -redex into a δ -redex or a ζ -redex, leaving the rest of the contraction to these two schemes. The δ scheme expands (i.e. unfolds) some instances of an abbreviation (but not necessarily all of them), so the binder remains in place after the expansion to allow other instances of the same abbreviation to be unfolded if necessary. The ζ scheme removes the binder of a fully expanded abbreviation (this can be related to COQ [Coq development team 2007] but the ζ scheme of COQ unfolds the abbreviation before removing its binder, which we do by invoking the δ scheme). The τ scheme makes type annotations eliminable up to reduction. In this way, we express the fact that these items are not strictly essential for reduction and typing. The v scheme is thought to contract the β -redex $(V_1).\lambda x:W$ when

$$\frac{}{E \Rightarrow_{we} E} \text{refl} \quad \frac{W_1 \Rightarrow W_2 \quad E_1 \Rightarrow_{we} E_2}{\lambda x:W_1.E_1 \Rightarrow_{we} \lambda x:W_2.E_2} \text{abst} \quad \frac{V_1 \Rightarrow V_2 \quad E_1 \Rightarrow_{we} E_2}{\delta x \leftarrow V_1.E_1 \Rightarrow_{we} \delta x \leftarrow V_2.E_2} \text{abbr}$$

$$\frac{V_1 \Rightarrow V_2 \quad E_1 \Rightarrow_{we} E_2}{(V_1).E_1 \Rightarrow_{we} (V_2).E_2} \text{appl} \quad \frac{W_1 \Rightarrow W_2 \quad E_1 \Rightarrow_{we} E_2}{\langle W_1 \rangle.E_1 \Rightarrow_{we} \langle W_2 \rangle.E_2} \text{cast}$$

Fig. 4. Weak parallel reduction rules on environments

$$\frac{T_1 \Rightarrow T_2}{E \vdash T_1 \Rightarrow T_2} \text{free} \quad \frac{E = C_1.\delta x \leftarrow V.C_2 \quad T_1 \Rightarrow T_2 \quad [x^+ \leftarrow V]_t T_2 = T}{E \vdash T_1 \Rightarrow T} \delta$$

Fig. 5. Environment-dependent parallel reduction rules

its two items are separated by an extraneous abbreviator (i.e. $\delta y \leftarrow V_2$). Without the ν -swap, the β -redex would be created only after removing this abbreviator by ζ -contraction; this means that the associated abbreviation should be completely unfolded before the removal. With the ν -swap, instead, we can obtain the β -redex without any unfolding and this is certainly more desirable in realistic use cases.

It is worth remarking how the full β -contraction is achieved in this calculus: the full β -contraction performs three atomic actions on the term $(V).\lambda x:W.T$: it removes the applicator, it removes the binder, it substitutes V for all occurrences of x in T . In $\lambda\delta$ special care is taken for having three different reduction schemes that take charge of these actions. The β scheme is responsible for removing the applicator (the binder is changed but it is not removed). The substitution is performed by invoking the δ scheme one or more times as long as x occurs in T . When the substitution is completed, the ζ scheme can be applied and the binder is removed.

As we see, the five reduction schemes are “orthogonal” or “primary” in the sense that a given redex belongs to just one scheme and therefore it reduces in a unique way. This means that we never have critical pairs. Here we are using “primary” as opposed to “auxiliary” of [Kamareddine and Bloo 2005b; 2005a]. Other primary or auxiliary reduction schemes might be considered as well.

The above reduction allows to define a weak parallel reduction on environments, which we use to prove the subject reduction results Theorem 9(1) and Theorem 2(1). This reduction is *weak* in the sense that it involves just the terms appearing in the environment items and not the environment items themselves.

DEFINITION 8 WEAK REDUCTION ON ENVIRONMENTS.

The relation $E_1 \Rightarrow_{we} E_2$ indicates one step of weak parallel reduction from the environment E_1 to the environment E_2 . Its rules are shown in Figure 4.

DEFINITION 9 ENVIRONMENT-DEPENDENT PARALLEL REDUCTION.

The relation $E \vdash T_1 \Rightarrow T_2$ indicates one step of environment-dependent parallel reduction from T_1 to T_2 . Its rules are shown in Figure 5 and the reduction steps are shown in Figure 6. Moreover the relation $E \vdash T_1 \Rightarrow^* T_2$ is the transitive closure of $\vdash \Rightarrow$ and the relation $E \vdash T_1 \Leftrightarrow^* T_2$ is the symmetric and transitive closure of $\vdash \Rightarrow$, that we call environment-dependent parallel conversion.

Also environment-dependent reduction is presented in its parallel form to ease the proof of confluence with itself (Theorem 3(3)). The effect of a step $E \vdash T_1 \Rightarrow T_2$ is

scheme	redex	reductum
δ -expansion	$C_1.\delta x \leftarrow V.C_2 \vdash T$	$\rightarrow_\delta [x^+ \leftarrow V]_t T$ if $x \in \text{FV}(T)$

Fig. 6. Environment-dependent reduction steps

to reduce a subset of the environment-free redexes appearing in T_1 and, optionally, to expand one or more instances of a global abbreviation stored in E .

We are aware that the δ rule of Figure 5 could be improved by using environment-dependent reduction in place of environment-free reduction in the second premise.

Finally we discard the widely used notation with the $=$ sign for the conversion relation because we feel that $=$ should be reserved for a generic equivalence relation. We could use $=_{\beta\delta\zeta\tau\nu}$ to indicate that conversion is equality up to the indicated reduction steps, but this notation does not make clear whether these steps are actually performed sequentially or in parallel.

We recall that a term is *normal* or *in normal form* [Barendregt 1993] when it can not be reduced. Here we use the following definition of a normal term.

DEFINITION 10 NORMAL TERMS.

The predicate $\text{nf}(E, T)$, stating that the term T is normal with respect to context-dependent parallel reduction $E \vdash \Rightarrow$, is defined as follows.

$$\text{nf}(E, T_1) \quad \text{iff} \quad \text{for each } T_2, E \vdash T_1 \Rightarrow T_2 \text{ implies } T_1 = T_2. \quad (3)$$

Here we are taking into account the fact that $E \vdash \Rightarrow$ is a reflexive relation.

We can also extend the normal form predicate to a list of terms meaning the conjunction of the predicate applied to each element of the list.

According to [Girard et al. 1989; Barendregt 1993] a term T is *strongly normalizable* if there is no infinite sequence of reduction steps starting from T .

DEFINITION 11 STRONGLY NORMALIZABLE TERMS.

The predicate $\text{sn}(E, T)$, stating that the term T is strongly normalizable with respect to context-dependent parallel reduction $E \vdash \Rightarrow$, is inductively defined by one clause that is a higher order rule:

$$\text{If for each } T_2, T_1 \neq T_2 \text{ and } E \vdash T_1 \Rightarrow^* T_2 \text{ imply } \text{sn}(E, T_2), \text{ then } \text{sn}(E, T_1) \quad (4)$$

Indeed if $E \not\vdash T_1 \Rightarrow^* T_2$ for all $T_2 \neq T_1$, then T_1 is normal and $\text{sn}(E, T_1)$ holds a fortiori. This is the base case of the structural induction defined by Rule (4).

Essentially we borrowed this definition from [Letouzey and Schwichtenberg 2004] but we had to take into account the fact that $E \vdash \Rightarrow^*$ is a reflexive relation. Moreover we would prefer to use $E \vdash \Rightarrow$ in place of $E \vdash \Rightarrow^*$ but $E \vdash \Rightarrow$ is not perfectly designed yet and some desirable properties fail to hold: for instance even if $E \vdash V_1 \Rightarrow V_2$ and $E \vdash T_1 \Rightarrow T_2$, it is not true that $E \vdash (V_1).T_1 \Rightarrow (V_2).T_2$.

We can also extend the strong normalization predicate to a list of terms meaning the conjunction of the predicate applied to each element of the list.

2.4 Native Type Assignment

In this subsection we present the native type system of $\lambda\delta$. Another type system, originally due to de Bruijn, is presented in Subsection 2.5.

The type judgement depends on the parameter defined below:

$$\begin{array}{c}
\frac{}{E \vdash_g \text{Sort}_h : \text{Sort}_{g(h)}} \text{sort} \\
\frac{E = C_1.\delta x \leftarrow V.C_2 \quad C_1 \vdash_g V : W}{E \vdash_g x : W} \text{def} \quad \frac{E = C_1.\lambda x:W.C_2 \quad C_1 \vdash_g W : V}{E \vdash_g x : W} \text{decl} \\
\frac{E \vdash_g V : W \quad E.\delta x \leftarrow V \vdash_g T : U}{E \vdash_g \delta x \leftarrow V.T : \delta x \leftarrow V.U} \text{abbr} \quad \frac{E \vdash_g W : V \quad E.\lambda x:W \vdash_g T : U}{E \vdash_g \lambda x:W.T : \lambda x:W.U} \text{abst} \\
\frac{E \vdash_g V : W \quad E \vdash_g T : \lambda x:W.U}{E \vdash_g (V).T : (V).\lambda x:W.U} \text{appl} \quad \frac{E \vdash_g T : W \quad E \vdash_g W : V}{E \vdash_g \langle W \rangle.T : \langle V \rangle.W} \text{cast} \\
\frac{E \vdash_g U_2 : W \quad E \vdash_g T : U_1 \quad E \vdash U_1 \leftrightarrow^* U_2}{E \vdash_g T : U_2} \text{conv}
\end{array}$$

Fig. 7. Native type assignment rules

DEFINITION 12 SORT HIERARCHY PARAMETER.

The sort hierarchy parameter is a function $g : \mathbb{N} \rightarrow \mathbb{N}$ that satisfies the strict monotonicity condition: $h < g(h)$ for all h .

The value $g(h)$ is the index of the sort that types Sort_h and the monotonicity of g is the simplest condition ensuring a loop-free type hierarchy of sorts. We use this condition to prove Theorem 10(6) (impossibility of typing a term with itself).

Notice that g is a total function but in the most general case a partial function should be used. This would allow sort hierarchies with top-level elements as the ones of many typed λ -calculi. Nevertheless this generalization is inconvenient since it complicates several theorems about typing without increasing the expressiveness of the calculus, in fact any sort hierarchy with top-level elements can be embedded in a sort hierarchy without top-level elements.

DEFINITION 13 NATIVE TYPE ASSIGNMENT.

The native type judgement has the form $E \vdash_g T : U$ where g is a sort hierarchy parameter. Its rules are shown in Figure 7.

Notice that the $\lambda\delta$ type judgement does not depend on the notion of a legal (i.e. well formed) context as it happens in other type systems (see for instance [Maietti and Sambin 2005]). This is because an unreferenced variable needs a legal declaration only if it is the formal argument of a function. This approach, which is closer to a realistic implementation of a type checker, has the technical benefit of simplifying the proofs of the properties of types because the mutual dependence between the type judgement and the legality judgement disappears.

The type policy of $\lambda\delta$ is that the type rules should be as close as possible to the usual rules of typed λ -calculus [Barendregt 1993]. The major modification lays in the type rule for abstraction, that is the composition of the usual type rules for λ and for Π . Here are the type rules for λ and for Π in the λ -cube.

$$\frac{\Gamma, x:A \vdash b : B \quad \Gamma \vdash (\Pi_{x:A}.B) : s}{\Gamma \vdash (\lambda_{x:A}.b) : (\Pi_{x:A}.B)} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x:A \vdash B : s_2}{\Gamma \vdash (\Pi_{x:A}.B) : s_2} \quad (5)$$

In $\lambda\delta$ we want to type an abstraction with an abstraction, therefore we remove the second premise of the first rule and the conclusion of the second rule. Then we make a single rule by combining the remaining judgements and by turning the Π into a λ . In addition we generalize the sorts s_1 and s_2 to arbitrary types. Moreover we

$$\begin{array}{c}
\frac{}{\text{st}_g(E, \text{Sort}_h) = \text{Sort}_{g(h)}} \text{sort} \\
\frac{E = C_1.\delta x \leftarrow V.C_2 \quad \text{st}_g(C_1, V) = W}{\text{st}_g(E, x) = W} \text{def} \quad \frac{E = C_1.\lambda x:W.C_2 \quad \text{st}_g(C_1, W) = V}{\text{st}_g(E, x) = W} \text{decl} \\
\frac{\text{st}_g(E.\delta x \leftarrow V, T) = U}{\text{st}_g(E, \delta x \leftarrow V.T) = \delta x \leftarrow V.U} \text{abbr} \quad \frac{\text{st}_g(E.\lambda x:W, T) = U}{\text{st}_g(E, \lambda x:W.T) = \lambda x:W.U} \text{abst} \\
\frac{\text{st}_g(E, T) = U}{\text{st}_g(E, (V).T) = (V).U} \text{appl} \quad \frac{\text{st}_g(E, V) = W \quad \text{st}_g(E, T) = U}{\text{st}_g(E, \langle V \rangle.T) = \langle W \rangle.U} \text{cast}
\end{array}$$

Fig. 8. Static type assignment rules

recently noticed that the second premise of the second rule becomes unnecessary. The rule we obtain at the end is Figure 7(abst). An important consequence of this rule, expressed by Theorem 10(1), is that a term and its type have the same functional structure, i.e. they take the same number of arguments when they are interpreted as functions, moreover the corresponding arguments of these functions have the same type. Stated in other words, a type fully determines the number of arguments taken by its inhabitants and the types of these arguments.

Figure 7(abbr) follows the scheme of Figure 7(abst) and is compatible with the commonly accepted Rule (6) for typing abbreviations found in [Coq development team 2007] since $B[x := A]$ and $(\delta_{x=A}.B)$ are $\delta\zeta$ -convertible. Notice that C does not need to be a sort in this rule.

$$\frac{\Gamma, x=A \vdash b : B \quad \Gamma \vdash A : C}{\Gamma \vdash (\delta_{x=A}.b) : B[x := A]} \quad (6)$$

In the spirit of Figure 7(abbr), the rule typing the application (Figure 7(appl) that we borrow from [Kamareddine et al. 1999]) does not apply any reduction at the level of types (like Rule (6) does, unfolding the abbreviation in the term B).

The technical benefit of this approach is that the reductional behavior of the type judgement is confined in the so-called “conversion rule”.

More sophisticated forms of typing, involving reductions in the environment (in the sense of Subsection 2.3) might be considered as well.

2.5 Static Type Assignment

The so-called *de Bruijn type assignment* (typ in [de Bruijn 1993] and in the Automath tradition) is a function introduced by de Bruijn as part of the type checking algorithm for the language Aut – 68. Here we define the analogous concept in $\lambda\delta$.

DEFINITION 14 STATIC TYPE ASSIGNMENT.

The partial function $\text{st}_g(E, T)$ evaluates the static type of a term T in the environment E , which depends on the parameter g . Its rules are shown in Figure 8.

The non-deterministic partial function $\text{st}_g^+(E, T)$ evaluates the composition of one or more applications of st_g to T in E . The ‘+’ recalls “one or more”.

Notice that this type is assigned by means of syntax-oriented rules that do not involve reduction, that is why we term this type *static* in this paper.

Obviously this feature makes the computation of the static type very fast. Another consequence is that the static type of a term inherits the binders and redexes of that term (i.e. it may have more binders and redexes but not less).

Besides being a very well established notion that also $\lambda\delta$ can deal with, the static type is relevant in this paper for two theoretical reasons. Firstly it allows to define an immersion of \mathbb{T} into \mathbb{E} that opens the road to a dualization of terms and environments (see Appendix B). Secondly it is used in Subsection 2.6 to justify the notion of arity, that plays an important role in connecting $\lambda\delta$ to $\lambda\rightarrow$.

2.6 Arity Assignment

The notion of arity [Nordström et al. 1990] (skeletons in [Barras 1996]) as a description of the functional structure of a term it is not strictly necessary in $\lambda\delta$ as well as the data type \mathbb{L} used to represent it (since arities can be encoded into terms). But both are useful from the technical standpoint. Arities are expected to provide for a connection between the terms of $\lambda\delta$ and the types of a suitable version of $\lambda\rightarrow$, they facilitate the proof of the strong normalization theorem (see Theorem 6(9)) and they speed up the proofs of the last three clauses of Theorem 10.

DEFINITION 15 ARITIES.

The set of arities is defined as follows:

$$\mathbb{L} \equiv (\mathbb{N}, \mathbb{N}) \mid \mathbb{L} \rightarrow \mathbb{L} \quad (7)$$

The arities of the form (h, k) are called nodes and are ordered pairs.

In the following, the variable L will always range over the data type \mathbb{L} .

The arity of a term T has the form $L \equiv L_1 \rightarrow L_2 \rightarrow \dots \rightarrow L_i \rightarrow (h, k)$ and it describes the following features of T :

- the *position* of T in the type hierarchy is the node (h, k) . By this we mean that iterating k times the static typing operation on T , we obtain a term whose rightmost item is Sort_h (this term exists as shown by Theorem 12(2));
- T is a function taking exactly i arguments (i.e. a function of arity i);
- for each j between 1 and i , the j -th argument of T must have arity L_j .

By looking at its shape, it should be clear that an arity is a type of the instance of $\lambda\rightarrow$ in which we take the nodes as basic types.

Notice that our arity of T , containing the position of all arguments of T , is more informative than the skeleton of [Barras 1996] that only records the position of T .

Also notice that we can not expect a term to have a unique position since each term at position (h, k) is also at position $(g(h), k + 1)$.⁵

In order to assign an arity to a declared variable we need a function connecting the arity of a term to the arity of its type. Here we present the *strict successor* function defined below but we are not positive on the fact that this is the best choice and we see two alternatives that might be considered as well.

The strict successor of a node depends on the sort hierarchy parameter g and the strict successor of an arity is a natural extension of the former. We also introduce the *strict sum* as the iterated composition of the strict successor.

DEFINITION 16 THE STRICT SUCCESSOR AND THE STRICT SUM.

The strict successor of the arity L , denoted by $L +_g 1$ is defined as follows:

⁵The converse is not true in general.

$$\frac{(h_1, k_1) +_g k = (h_2, k_2) +_g k}{(h_1, k_1) =_g (h_2, k_2)} \text{node} \quad \frac{L_1 =_g L_2 \quad L_3 =_g L_4}{L_1 \rightarrow L_3 =_g L_2 \rightarrow L_4} \text{impl}$$

Fig. 9. Level equality rules

$$\left\{ \begin{array}{l} (h, 0) +_g 1 \quad \equiv \quad (g(h), 0) \\ (h, k + 1) +_g 1 \quad \equiv \quad (h, k) \\ (L_1 \rightarrow L_2) +_g 1 \quad \equiv \quad L_1 \rightarrow (L_2 +_g 1) \end{array} \right. \quad (8)$$

The strict sum $L +_g k$ is the composition of k strict successors applied to L .

$$\left\{ \begin{array}{l} L +_g 0 \quad \equiv \quad L \\ L +_g (k + 1) \quad \equiv \quad (L +_g k) +_g 1 \end{array} \right. \quad (9)$$

We may think of the type hierarchy induced by the parameter g as an oriented graph in which the arcs are drawn from each node L to its strict successor $L +_g 1$.

Coming now to the problem of defining the level (class in [Barras 1996]) of a node in the type hierarchy graph, i.e. the height of this node from a reference point, we observe that this notion can not be given in absolute terms (as it happens in the type hierarchies with top-level elements or bottom-level elements) because in our case the graph can be disconnected so no node can be taken as a global reference point. The best we can do is to define what it means for two nodes L_1 and L_2 to be *at the same level* by saying that they must have the same height relatively to a third node L_3 to which they are both connected.

So we say that the nodes L_1 and L_2 are at the same level in the type hierarchy if there exists k such that $L_1 +_g k = L_2 +_g k$ and we express this concept as follows.

DEFINITION 17 LEVEL QUALITY.

The level equality predicate $L_1 =_g L_2$ is defined by the rules in Figure 9.

Notice that $=_g$ is an equivalence relation and that $(h, k) =_g (g(h), k + 1)$ in fact $(h, k) +_g (k + 1) = (g(h), 0) = (g(h), k + 1) +_g (k + 1)$.

Formally the levels of the type hierarchy are the equivalence classes of $=_g$.

If we chose $g(h) \equiv h + 1$, the levels of the corresponding type hierarchy are isomorphic to the integer numbers, as shown by Theorem 13, and the integer number associated to the equivalence class containing the node (h, k) is $h - k$. This result is consistent with the intuition according to which the type hierarchy of $\lambda\delta$ has an infinite sequence of levels both above and below any reference point.⁶

It is important to remark that the decidability of the predicate $=_g$ depends on the choice of the parameter g . This predicate is undecidable in general but it is decidable for some choices of g , for instance for the one above.

Now we have all the ingredients to define the arity assignment.

DEFINITION 18 ARITY ASSIGNMENT.

The arity assignment predicate is $E \vdash_g T \triangleright L$ and means that the term T has arity L in the context E with respect to g . Its rules are given in Figure 10.

⁶If we define $(h, k) +_g z \equiv (h, k - z)$ when $z < 0$, then the function $z \mapsto L +_g z$ from the integer numbers to \mathbb{L} is injective with respect to $=_g$ in the sense that $L +_g z_1 =_g L +_g z_2$ implies $z_1 = z_2$. This fact is not proved in [Guidi 2007a] yet.

$$\begin{array}{c}
\frac{E \vdash_g T \triangleright L_1 \quad L_1 =_g L_2}{E \vdash_g T \triangleright L_2} \text{ repl} \quad \frac{}{E \vdash_g \text{Sort}_h \triangleright (h, 0)} \text{ sort} \\
\frac{E = C_1.\delta x \leftarrow V.C_2 \quad C_1 \vdash_g V \triangleright L}{E \vdash_g x \triangleright L} \text{ def} \quad \frac{E = C_1.\lambda x:W.C_2 \quad D \vdash_g W \triangleright L +_g 1}{E \vdash_g x \triangleright L} \text{ decl} \\
\frac{E \vdash_g V \triangleright L_1 \quad E.\delta x \leftarrow V \vdash_g T \triangleright L_2}{E \vdash_g \delta x \leftarrow V.T \triangleright L_2} \text{ abbr} \quad \frac{E \vdash_g V \triangleright L_1 +_g 1 \quad E.\lambda x:W \vdash_g T \triangleright L_2}{E \vdash_g \lambda x:W.T \triangleright L_1 \rightarrow L_2} \text{ abst} \\
\frac{E \vdash_g V \triangleright L_1 \quad E \vdash_g T \triangleright L_1 \rightarrow L_2}{E \vdash_g (V).T \triangleright L_2} \text{ appl} \quad \frac{E \vdash_g W \triangleright L +_g 1 \quad E \vdash_g T \triangleright L}{E \vdash_g \langle W \rangle.T \triangleright L} \text{ cast}
\end{array}$$

Fig. 10. Arity assignment rules

In this paper we assign the arity up to level equality, but we suspect that other (more desirable) solutions are possible as well.

2.7 Domain-Based Preorders on Environments

We recall that a variable occurrence x is a placeholder for a member of a given subset of terms, which is called the *domain* of x . In our case if x is bound in the environment $E_1 \equiv C.\lambda x:W$ then x stands for any term of type W in C so its domain is $\mathcal{D}_1 \equiv \{T \mid C \vdash_g T : W\}$. On the other hand if x is bound in the environment $E_2 \equiv C.\delta x \leftarrow V$ then x stands only for V so its domain is $\mathcal{D}_2 \equiv \{T \mid T = V\}$.

If we now assume $C \vdash_g V : W$, we see that $\mathcal{D}_2 \subseteq \mathcal{D}_1$ and we are led to define the following preorder \preceq_g on environments such that $E_2 \preceq_g E_1$ holds.

DEFINITION 19 DOMAIN-BASED PREORDER ON ENVIRONMENTS.

The relation $E_2 \preceq_g E_1$ holds when the environments E_2 and E_1 bind the same variables and for each of these variables, its domain in E_2 is contained in its domain in E_1 .⁷ The rules of this relation are given below:

- (sort) $\text{Sort}_h \preceq_g \text{Sort}_h$;
- (compatibility) if $C_2 \preceq_g C_1$ then $\lambda x:W.C_2 \preceq_g \lambda x:W.C_1$ and $\delta x \leftarrow V.C_2 \preceq_g \delta x \leftarrow V.C_1$ and $(V).C_2 \preceq_g (V).C_1$ and $\langle W \rangle.C_2 \preceq_g \langle W \rangle.C_1$;
- (abst) if $C_2 \preceq_g C_1$ and $C_2 \vdash_g V : W$ and $C_1 \vdash_g V : W$ then $C_2.\delta x \leftarrow V \preceq_g C_1.\lambda x:W$.

The preorder \preceq_g is an auxiliary notion we use to prove the subject reduction property of the native type assignment, Theorem 9(1), in the case of the β -contraction because of the shapes of the β -reductum (Figure 3), of Figure 7(abst) and of Figure 7(abbr). In fact we know that the calculi in which the β -reductum exploits an explicit substitution in place of an abbreviation, do not need this apparatus.

If we relax the minor premises of Definition 19(abst) by expressing them in terms of the arity assignment, we obtain the preorder \sqsubseteq_g defined below:

DEFINITION 20 RELAXED DOMAIN-BASED PREORDER ON ENVIRONMENTS.

The relation $E_2 \sqsubseteq_g E_1$ is defined like $E_2 \preceq_g E_1$ but Definition 19(abst) is replaced by the following axiom:

- (abst) if $C_2 \sqsubseteq_g C_1$ and $C_2 \vdash_g V \triangleright L$ and $C_1 \vdash_g W \triangleright L +_g 1$ then $C_2.\delta x \leftarrow V \sqsubseteq_g C_1.\lambda x:W$.

⁷In [Guidi 2007a] we axiomatized the relation “ $E_1 \succeq_g E_2$ ” rather than “ $E_2 \preceq_g E_1$ ”.

We use this preorder as an auxiliary notion to prove the subject reduction property of the arity assignment, Theorem 2(1), in the case of the β -contraction because of the shapes of the β -reductum (Figure 3), of Figure 10(abst) and of Figure 10(abbr). We stress that \sqsubseteq_g is undecidable in general because it involves $=_g$.

Notice that Theorem 10(3) states that $E_2 \preceq_g E_1$ implies $E_2 \sqsubseteq_g E_1$ but we argue from Theorem 14 that the converse does not hold in general.

3. THE THEORY OF $\lambda\delta$

In this section we present the main properties of the notions we introduced in Section 2. In particular we give the results on arities (Subsection 3.1), on reduction and conversion (Subsection 3.2), on native types (Subsection 3.3) and on static types (Subsection 3.4). Notice that here we are forced to order the topics in a slightly different way with respect to Section 2 because we want to follow the dependency graph of the theorems we present. In Subsection 3.5 we give some theorems about concrete terms and instances of the parameter g having interesting properties.

3.1 Results on the Arity Assignment

The arity assignment is an auxiliary notion in $\lambda\delta$, that we mainly introduced just to reduce the strong normalization of $\lambda\delta$ to that of $\lambda\rightarrow$. Furthermore the *replacement* arity assignment rule, Figure 10(repl), its not satisfactory because it involves the level equality predicate, which is undecidable in general. For these reasons we prefer not to insist on the results on arities and we just give some examples below.

THEOREM 1 MAIN PROPERTIES OF ARITIES.

- (1) *(every node is inhabited)*
For all h, k there exist C, T such that $C \vdash_g T \triangleright (h, k)$.
- (2) *(uniqueness of arity up to level equality)*
If $C \vdash_g T \triangleright L_1$ and $C \vdash_g T \triangleright L_2$ then $L_1 =_g L_2$.
- (3) *(substitution in focalized terms preserves the arity)*
If $C_1 \vdash_g T_1 \triangleright L$ and $C_1 = E.\delta x \leftarrow V.E'$ and $[x^+ \leftarrow V]_f(C_1, T_1) = (C_2, T_2)$ then $C_2 \vdash_g T_2 \triangleright L$.
- (4) *(monotonicity of the arity assignment with respect to \sqsubseteq_g)*
If $C_1 \vdash_g T \triangleright L$ and $C_2 \sqsubseteq_g C_1$ then $C_2 \vdash_g T \triangleright L$.

PROOF. Clause (1) is proved by induction on k . Clause (2) is proved by induction on the first premise and by cases on the second premise. Clause (3) is proved by induction on the first premise and by cases on the third premise. Clause (4) is proved by induction on the first premise with some invocations of Clause (2). \square

The subject reduction property of the arity assignment is proved by the theorem below. The main part of the proof is in the base case, where a single step of environment-free parallel reduction is considered. The possibility to reduce some terms inside the environment is essential here. The general case is just a corollary. As a consequence, the level of a term in the type hierarchy is preserved by reduction.

THEOREM 2 SUBJECT REDUCTION.

- (1) *(base case)*
If $C_1 \vdash_g T_1 \triangleright L$ and $C_1 \Rightarrow_{we} C_2$ and $T_1 \Rightarrow T_2$ then $C_2 \vdash_g T_2 \triangleright L$.

- (2) (*general case without the reduction in the environment*)
 If $C \vdash T_1 \Rightarrow^* T_2$ and $C \vdash_g T_1 \triangleright L$ then $C \vdash_g T_2 \triangleright L$.

PROOF. Clause (1) is proved by double induction on the first and third premise. In the case of Figure 10(abbr) against Figure 2(δ) we exploit Theorem 1(3), and in the case of Figure 10(appl) against Figure 2(β) we exploit Theorem 1(4). Clause (2) is proved by induction on the first premise via the previous clause. \square

3.2 The Results on Reduction and Conversion

The most relevant properties of reduction and conversion are listed below.

THEOREM 3 MAIN PROPERTIES OF REDUCTION AND CONVERSION.

- (1) (*confluence of \Rightarrow with strict substitution*)
 If $T_1 \Rightarrow T_2$ and $[x^+ \leftarrow W_1]_t T_1 = U_1$ and $W_1 \Rightarrow W_2$ then $U_1 \Rightarrow T_2$ or there exists U_2 such that $U_1 \Rightarrow U_2$ and $[x^+ \leftarrow W_2]_t T_2 = U_2$.
- (2) (*confluence of \Rightarrow with itself: Church-Rosser property*)
 If $T_0 \Rightarrow T_1$ and $T_0 \Rightarrow T_2$ then there exists T such that $T_1 \Rightarrow T$ and $T_2 \Rightarrow T$.
- (3) (*confluence of $\vdash \Rightarrow^*$ with itself: Church-Rosser property*)
 If $C \vdash T_0 \Rightarrow^* T_1$ and $C \vdash T_0 \Rightarrow^* T_2$ then there exists T such that $C \vdash T_1 \Rightarrow^* T$ and $C \vdash T_2 \Rightarrow^* T$.
- (4) (*thinning of the applicator for $\vdash \Leftrightarrow^*$*)
 If $C \vdash T_1 \Leftrightarrow^* T_2$ then $C \vdash (V).T_1 \Leftrightarrow^* (V).T_2$.
- (5) (*compatibility for $\vdash \Leftrightarrow^*$: first operand*)
 If $C \vdash V_1 \Leftrightarrow^* V_2$ then $C \vdash \lambda x:V_1.T \Leftrightarrow^* \lambda x:V_2.T$ and $C \vdash (V_1).T \Leftrightarrow^* (V_2).T$ and $C \vdash \delta x \leftarrow V_1.T \Leftrightarrow^* \delta x \leftarrow V_2.T$ and $C \vdash \langle V_1 \rangle.T \Leftrightarrow^* \langle V_2 \rangle.T$.
- (6) (*compatibility for $\vdash \Leftrightarrow^*$: second operand*)
 If $C.\lambda x:V \vdash T_1 \Leftrightarrow^* T_2$ then $C \vdash \lambda x:V.T_1 \Leftrightarrow^* \lambda x:V.T_2$; if $C.\delta x \leftarrow V \vdash T_1 \Leftrightarrow^* T_2$ then $C \vdash \delta x \leftarrow V.T_1 \Leftrightarrow^* \delta x \leftarrow V.T_2$.
- (7) (*generation lemma on abstraction for $\vdash \Leftrightarrow^*$*)
 If $C \vdash \lambda x:V_1.T_1 \Leftrightarrow^* \lambda x:V_2.T_2$ then $C \vdash V_1 \Leftrightarrow^* V_2$ and for all V , $C.\lambda x:V \vdash T_1 \Leftrightarrow^* T_2$.
- (8) (*η -conversion for the terms that convert to λ -abstractions*)
 If $C \vdash T \Leftrightarrow^* \lambda x:W.U$ and $C \vdash V \Leftrightarrow^* W$ and $x \notin \text{FV}(T)$ then $C \vdash \lambda x:V.(x).T \Leftrightarrow^* T$.

PROOF. Clause (1) is proved by induction on the first premise and by cases on the second premise. Clause (2) is proved by induction on T_0 and by cases on the two premises. Here we must assume that the inductive hypothesis holds for all proper subterms of T_0 . Clause (3) is a standard corollary of the previous clause, proved using the “strip lemma” [Barendregt 1993]. Clauses (4), (5), (6) are immediate. Clause (7) is proved by induction on the premise with the standard technique used for generation lemmas [Barendregt 1993]. Clause (8) is a corollary of clause (4). \square

The main result on reduction is Church-Rosser property, while the main result on conversion is its generation lemma on abstraction: a desirable property mentioned in [van Daalen 1980]. The other properties, stating that conversion is a congruence, are referenced in Appendix A.

What follows is a classification of the normal terms having an arity:

THEOREM 4 THE NORMAL TERMS WITH AN ARITY.

If $C \vdash_g T \triangleright L$ and $\text{nf}(C, T)$ then there exist \bar{V}, U, W, x, h such that:

- (1) $T = \lambda x:W.U$ and $\text{nf}(C, W)$ and $\text{nf}(C.\lambda x:W, U)$ or
- (2) $T = \text{Sort}_h$ or
- (3) $T = (\bar{V}).x$ and $\text{nf}(C, \bar{V})$ and $\text{nf}(C, x)$.

PROOF. By induction on the first premise and by cases on the second premise. \square

The strong normalization theorem outlined below, stating that every term with an arity is strongly normalizable, is one of the relevant results of the present paper.

If we consider the connections between $\lambda\delta$ and $\lambda\rightarrow$ that we briefly sketched in Subsection 2.6, it should not be a surprise that the proof of strong normalization proposed by Tait for $\lambda\rightarrow$ can be adapted for $\lambda\delta$. Namely both the definition of the strong reducibility candidates and the overall proof method are the same.

Our formalization follows essentially the version of Tait's proof reported by [Loader 1998]. Other references we considered are [Letouzey and Schwichtenberg 2004; Girard et al. 1989; Cescutti 2001; van Oostrom 2002]. The main difference with respect to [Loader 1998] is that we can use abbreviations in place of explicit substitutions because of the shape of our β -reductum (see Figure 2(β)).

DEFINITION 21 THE STRONG REDUCIBILITY CANDIDATES.

The subset of the focalized terms that are strong reducibility candidates of arity L (with respect to the parameter g) is here denoted by $[L]_g$ and it is defined below.

$$\left\{ \begin{array}{ll} (E, T) \in [(h, k)]_g & \text{iff } E \vdash_g T \triangleright (h, k) \text{ and } \text{sn}(E, T) \\ (E, T) \in [L_1 \rightarrow L_2]_g & \text{iff } E \vdash_g T \triangleright L_1 \rightarrow L_2 \text{ and for each } C, C_1, C_2, V, \\ & (C, V) \in [L_1]_g \text{ and } C = C_1.E.C_2 \text{ imply } (C, (V).T) \in [L_2]_g \end{array} \right. \quad (10)$$

Notice that the possibility of exchanging the binders of the environment C is silently assumed at least in Theorem 6(5) below (see [Loader 1998]). Thus Definition 21 must be rephrased carefully when binders are referenced by position instead of by name (i.e with de Bruijn indexes) as in [Guidi 2007a] (see Definition 33).

We also define a version of the relaxed preorder on environments (Definition 20) for use with the strong reducibility candidates, which we need in Theorem 6(8).

DEFINITION 22 RELAXED PREORDER ON ENVIRONMENTS FOR CANDIDATES.

The relation $E_2 \sqsubseteq_g^{\text{rc}} E_1$ is defined like $E_2 \sqsubseteq_g E_1$ but Definition 20(abst) is replaced by the axiom below. The notation "rc" stands for "reducibility candidates".

—(abst) if $C_2 \sqsubseteq_g^{\text{rc}} C_1$ and $(C_2, V) \in [L]_g$ and $(C_1, W) \in [L +_g 1]_g$ then $C_2.\delta x \leftarrow V \sqsubseteq_g^{\text{rc}} C_1.\lambda x:W$.

Here are the main results on the preorder we just defined:

THEOREM 5 MAIN PROPERTIES OF THE RELATION $\sqsubseteq_g^{\text{rc}}$.

- (1) (the preorder for candidates implies the relaxed preorder)
If $C_2 \sqsubseteq_g^{\text{rc}} C_1$ then $C_2 \sqsubseteq_g C_1$.
- (2) (monotonicity of the arity assignment with respect to $\sqsubseteq_g^{\text{rc}}$)
If $C_2 \sqsubseteq_g^{\text{rc}} C_1$ and $C_1 \vdash_g T \triangleright L$ then $C_2 \vdash_g T \triangleright L$.

PROOF. Clause (1) is easily proved by induction on its premise. Clause (2) is a corollary of the previous clause and of Theorem 1(4). \square

The strong normalization property, which we write as $C \vdash_g T \triangleright L$ implies $\text{sn}(g, T)$, is not proved as is, but is derived from a number of lemmas, which must be suitably generalized in order to be proved.

THEOREM 6 MAIN PROPERTIES OF THE STRONGLY NORMALIZABLE TERMS.

- (1) *(normal terms are strongly normalizable)*
If $\text{nf}(C, T)$ then $\text{sn}(C, T)$.
- (2) *(candidate type cast)*
If $(C, (\bar{V}).V) \in [L +_g 1]_g$ and $(C, (\bar{V}).T) \in [L]_g$ then $(C, (\bar{V}).\langle V \rangle.T) \in [L]_g$.
- (3) *(candidate reference to abbreviation)*
If $(C, (\bar{V}).V) \in [L]_g$ and $C = D.\delta x \leftarrow V.D'$ then $(C, (\bar{V}).x) \in [L]_g$.
- (4) *(candidate reference to abstraction)*
If $C \vdash_g (\bar{V}).x \triangleright L$ and $\text{nf}(C, x)$ and $\text{sn}(C, \bar{V})$ then $(C, (\bar{V}).x) \in [L]_g$.
- (5) *(candidates are strongly normalizable)*
If $(C, T) \in [L]_g$ then $\text{sn}(C, T)$.
- (6) *(candidate abbreviation)*
If $(C.\delta x \leftarrow V, (\bar{V}).T) \in [L_2]_g$ and $(C, V) \in [L_1]_g$ then $(C, (\bar{V}).\delta x \leftarrow V.T) \in [L_2]_g$.
- (7) *(candidate β -redex)*
If $(C, (\bar{V}).\delta x \leftarrow V.T) \in [L_2]_g$ and $(C, V) \in [L_1]_g$ and $(C, W) \in [L_1 +_g 1]_g$ then $(C, (\bar{V}).\langle V \rangle.\lambda x:W.T) \in [L_2]_g$.
- (8) *(terms with an arity are candidates, general case)*
If $C_1 \vdash_g T \triangleright L$ and $E = C_1.D$ and $C_2 \sqsubseteq_g^{\text{rc}} E$ then $(C_2, T) \in [L]_g$.
- (9) *(terms with an arity are candidates)*
If $C \vdash_g T \triangleright L$ then $(C, T) \in [L]_g$.

PROOF. Clause (1) is immediate. Clauses (2), (3), (4) and (5) are proved by induction on L . Notice however that clauses (4) and (5) must be proved simultaneously. Clauses (6) and (7) are proved by induction on L_2 by invoking clause (5). Clause (8) is proved by induction on its first premise and by cases on its third premise; here we invoke the clauses (2), (3), (4), (6), (7) with \bar{V} as the empty list \circ but this assumption is too weak to prove the clauses themselves; in the proof we also invoke Theorem 5(2). Clause (9) follows from the previous clause. \square

The fact that every term with an arity is strongly normalizing follows from the composition of Theorem 6(9) (the main result) and Theorem 6(5), but notice that the converse is not true in general as we imply from Theorem 6(1) and Theorem 15.

3.3 Results on the Native Type Assignment

The first result about the type system is the generation (i.e. inversion) lemma, whose aim is to invert the type assignment rules of Definition 13.

THEOREM 7 GENERATION LEMMA FOR NATIVE TYPE ASSIGNMENT.

- (1) *(generation lemma on sorts)*
If $C \vdash_g \text{Sort}_h : T$ then $C \vdash \text{Sort}_{g(h)} \Leftrightarrow^* T$.

- (2) (*generation lemma on bound references*)
 If $C \vdash_g x : T$ then there exist E, E', V, U such that $C \vdash U \Leftrightarrow^* T$ and $C = E.\delta x \leftarrow V.E'$ and $E \vdash_g V : U$ or there exist E, E', V, U such that $C \vdash V \Leftrightarrow^* T$ and $C = E.\lambda x : V.E'$ and $E \vdash_g V : U$.
- (3) (*generation lemma on abbreviations*)
 If $C \vdash_g \delta x \leftarrow V.U_1 : T$ then there exist U_2, U such that $C \vdash \delta x \leftarrow V.U_2 \Leftrightarrow^* T$ and $C \vdash_g V : U$ and $C.\delta x \leftarrow V \vdash_g U_1 : U_2$.
- (4) (*generation lemma on abstractions*)
 If $C \vdash_g \lambda x : V.U_1 : T$ then there exist U_2, U such that $C \vdash \lambda x : V.U_2 \Leftrightarrow^* T$ and $C \vdash_g V : U$ and $C.\lambda x : V \vdash_g U_1 : U_2$.
- (5) (*generation lemma on applications*)
 If $C \vdash_g (V_1).U_1 : T$ then there exist V_2, U_2 such that $C \vdash (V_1).\lambda x : V_2.U_2 \Leftrightarrow^* T$ and $C \vdash_g U_1 : \lambda x : V_2.U_2$ and $C \vdash_g V_1 : V_2$.
- (6) (*generation lemma on type annotations*)
 If $C \vdash_g \langle V \rangle.U : T$ then there exists V_0 such that $C \vdash \langle V_0 \rangle.V \Leftrightarrow^* T$ and $C \vdash_g U : V$ and $C \vdash_g V : V_0$.

PROOF. All clauses are proved by induction on the premise with the standard technique used to prove generation lemmas in general [Barendregt 1993]. \square

Some important properties of the native type assignment are listed below.

THEOREM 8 MAIN PROPERTIES OF NATIVE TYPE ASSIGNMENT.

- (1) (*thinning preserves type*)
 If $C_2 \vdash_g T_1 : T_2$ and $C_1 = D'.C_2.D''$ then $C_1 \vdash_g T_1 : T_2$.
- (2) (*correctness of types*)
 If $C \vdash_g T_1 : T_2$ then there exists T_3 such that $C \vdash_g T_2 : T_3$.
- (3) (*uniqueness of types up to conversion*)
 If $C \vdash_g T : T_1$ and $C \vdash_g T : T_2$ then $C \vdash T_1 \Leftrightarrow^* T_2$.
- (4) (*substitution in focalized terms preserves the type*)
 If $C_1 \vdash_g T_1 : T$ and $[x^+ \leftarrow V]_f(C_1, T_1) = (C_2, T_2)$ and $C_1 = E.\delta x \leftarrow V.E'$ then $C_2 \vdash_g T_2 : T$.
- (5) (*substitution in terms preserves the type*)
 If $C \vdash_g T_1 : T$ and $[x^+ \leftarrow V]_t T_1 = T_2$ and $C = E.\delta x \leftarrow V.E'$ then $C \vdash_g T_2 : T$.
- (6) (*substitution in environments preserves the type*)
 If $C_1 \vdash_g T : T_0$ and $[x^+ \leftarrow V]_e C_1 = C_2$ and $C_1 = E.\delta x \leftarrow V.E'$ then $C_2 \vdash_g T : T_0$.
- (7) (*monotonicity of the type assignment with respect to \preceq_g*)
 If $C_1 \vdash_g T_1 : T_2$ and $C_2 \preceq_g C_1$ then $C_2 \vdash_g T_1 : T_2$.
- (8) (*type checking implies type inference*)
 If $C \vdash_g T : V$ then there exists U such that $C \vdash_g \langle V \rangle.T : U$.

PROOF. Clause (1) is proved by induction on the first premise. The proof of clauses (2) and (3) is by induction on their first premise and contains invocations of Theorem 7 and of clause (1). Clause (4) is proved by double induction on the first two premises and by invoking the previous clauses. The statements (5) and (6) are mutually recursive so we prove them as corollaries of clause (4). Clause (7) is proved by induction on the first premise. Clause (8) is a corollary of clause (2). \square

A consequence of Theorem 7(6) is that if $\langle W \rangle.T$ is typable in E then T has type W in E . The converse also holds by Theorem 8(8) and this implies that in $\lambda\delta$, type checking can be expressed in terms of type inference [Barendregt 1993].

Theorem 8(7) is the most relevant result about the preorder \preceq_g .

The subject reduction of $\lambda\delta$ is one of the main results we are presenting in this paper. The main part of the proof is concentrated in the base case, where a single step of environment-free parallel reduction is considered. The possibility to reduce some terms appearing inside the environment is essential here (see [Kamareddine et al. 1999]). The general case is just a simple corollary.

THEOREM 9 SUBJECT REDUCTION AND COROLLARIES.

- (1) *(base case)*
If $C_1 \vdash_g T : T_2$ and $C_1 \Rightarrow_{we} C_2$ and $T \Rightarrow T_1$ then $C_2 \vdash_g T_1 : T_2$.
- (2) *(general case without the reduction in the environment)*
If $C \vdash T \Rightarrow^* T_1$ and $C \vdash_g T : T_2$ then $C \vdash_g T_1 : T_2$.
- (3) *(inverse of type preservation by thinning)*
If $C_1 \vdash_g T : T_1$ and $C_1 = D'.C_2.D''$ then there exists T_2 such that $C_1 \vdash T_2 \Leftrightarrow^* T_1$ and $C_2 \vdash_g T : T_2$.
- (4) *(type reduction)*
If $C \vdash_g T : T_1$ and $C \vdash T_1 \Rightarrow^* T_2$ then $C \vdash_g T : T_2$.
- (5) *(subject conversion: first case)*
If $C \vdash_g U_1 : T_1$ and $C \vdash_g U_2 : T_2$ and $C \vdash U_1 \Leftrightarrow^* U_2$ then $C \vdash T_1 \Leftrightarrow^* T_2$.
- (6) *(subject conversion: second case)*
If $C \vdash_g U_1 : T_1$ and $C \vdash_g U_2 : T_2$ and $C \vdash U_1 \Leftrightarrow^* U_2$ then $C \vdash_g U_1 : T_2$.

PROOF. Clause (1) is proved by induction on the first premise and by cases on the third premise with frequent invocations of Theorem 7, Theorem 8(1) and Theorem 8(2). In the case of Figure 7(abbr) against Figure 2(δ) we exploit Theorem 8(5), and in the case of Figure 7(appl) against Figure 2(β) we exploit Theorem 8(7). Clause (2) is corollary of the previous clause proved by induction on the first premise. Clause (3) is proved by induction on the first premise. Clauses (4) and (6) are corollaries of Theorem 8(2). Clause (5) is a corollary of Theorem 8(3). \square

We would like to stress that the proof of the subject reduction is more difficult in $\lambda\delta$ than in the λ -cube because in $\lambda\delta$ we can not assume that the type of the type of a term is a sort (as it is often done in λ -cube).

With Theorem 9(1) we avoid the simultaneous induction with which many authors, including [Kamareddine et al. 1999], prove the results like Theorem 9(2). Notice that Theorem 9(6) is stated as a desired property in [van Daalen 1980].

Some properties of the type system are proved more easily invoking arities because arities are assigned up to level equality instead of up to conversion and level equality is easier to manage being defined by simpler rules. The other rules of the arity assignment have the same complexity of the corresponding rule for the types.

THEOREM 10 SOME PROPERTIES OF TYPES PROVED USING ARITIES.

- (1) *(typed terms have an arity)*
If $C \vdash_g T_1 : T_2$ then there exists L such that $C \vdash_g T_1 \triangleright L$ and $C \vdash_g T_2 \triangleright L +_g 1$.

- (2) (*typed terms are strongly normalizable*)
If $C \vdash_g T : U$ then $\text{sn}(C, T)$.
- (3) (*the preorder on environments implies the relaxed preorder*)
If $C_2 \preceq_g C_1$ then $C_2 \sqsubseteq_g C_1$.
- (4) (*abstraction is predicative*)
If $C \vdash_g \lambda x:V.T : U$ then $C \not\vdash U \Leftrightarrow^* V$.
- (5) (*abstraction is not absorbent*)
If $C \vdash_g \lambda x:V.T : U_1$ and $C.\lambda x:V \vdash_g T : U_2$ and $x \notin \text{FV}(U_2)$ then $C \not\vdash U_1 \Leftrightarrow^* U_2$.
- (6) (*terms can not be typed with themselves*)
If $C \vdash_g T : U$ then $C \not\vdash U \Leftrightarrow^* T$.

PROOF. Clause (1) is a consequence of Theorem 2, it is proved by induction on its premise and it is a prerequisite of the other clauses. In particular clause (2) is a corollary of Theorem 6(9) and Theorem 6(5), Clause (3) is proved by induction on its premise by invoking Theorem 1(4). clause (4) invokes Theorem 7(4), clause (5) invokes Theorem 8(2), and clause (6) uses the strict monotonicity condition of the sort hierarchy parameter g (see Definition 12). \square

Notice that Theorem 10(1) includes our version of the theorem stating that the level of a term and the level of its type differ in one application of the successor function (originally proved by de Bruijn for his calculi).

Theorem 10(4) states that a term constructed by abstraction never belongs to the abstraction domain (i.e. the class of the terms typed by V in this case). Moreover Theorem 10(5) states that in $\lambda\delta$ there is no term $*$ for which, in standard notation:

$$\frac{\Gamma \vdash A : * \quad \Gamma, x:A \vdash B : *}{\Gamma \vdash (\lambda_{x:A}.B) : *} \quad (11)$$

We stress that Theorem 10(4) and Theorem 10(5) are expected properties of the λ -abstraction, which hold in every typed λ -calculus.

The decidability results we present below are a consequence of Theorem 10(2).

THEOREM 11 MAIN DECIDABILITY RESULTS.

- (1) (*convertibility of typed terms is decidable*)
If $C \vdash_g U_1 : T_1$ and $C \vdash_g U_2 : T_2$ then $C \vdash U_1 \Leftrightarrow^* U_2$ or $C \not\vdash U_1 \Leftrightarrow^* U_2$.
- (2) (*type inference is decidable*)
For all C, T_1 there exists T_2 such that $C \vdash_g T_1 : T_2$ or for all $T_2, C \not\vdash_g T_1 : T_2$.

PROOF. Clause (1) is a standard consequence of Theorem 10(2) and Theorem 3(3). Clause (2) is proved by induction on the focalized term (C, T_1) using Theorem 7, Theorem 8(2), Theorem 9(2) and the previous clause. We assume that the inductive hypothesis holds for all proper subterms of (C, T_1) (intended as the term $C.T_1$). Moreover we consider $(E.\lambda x:W, T)$ and $(E.\delta x \leftarrow V, T)$ as subterms of $(E, \lambda x:W.T)$ and $(E, \delta x \leftarrow V.T)$ respectively (because of Figure 7(abst) and Figure 7(abbr)). \square

Notice that by Theorem 7(6) and Theorem 8(8), type checking is also decidable.

$$\frac{k_1 + h_2 = k_2 + h_1}{(h_1, k_1) =_z (h_2, k_2)} \text{node} \quad \frac{L_1 =_z L_2 \quad L_3 =_z L_4}{L_1 \rightarrow L_3 =_z L_2 \rightarrow L_4} \text{impl}$$

Fig. 11. Integer level equality rules

3.4 Results on the Static Type Assignment

The main results about $\text{st}_g(C, \cdot)$ are listed below.

THEOREM 12 MAIN PROPERTIES OF THE STATIC TYPE.

- (1) (a typable term is typed by its static type)
If $C \vdash_g U : T_1$ and $\text{st}_g(C, U) = T_2$ then $C \vdash_g U : T_2$.
- (2) (the iterated static type yields a term that can be seen as an environment)
If $\text{st}_g(C, T_1) = T$ then there exists T_2 such that $\text{st}_g^+(C, T_1) = T_2$ and $\text{env}(T_2)$.

PROOF. Clause (1) is proved by induction on the first premise and by cases on the second premise. While considering Figure 7(appl) and Figure 7(cast), we invoke Theorem 7, Theorem 8(2), Theorem 8(3) and Theorem 9(6). Clause (2) is easily proved by induction on its premise. \square

Theorem 12(1) shows that the static type is indeed a type if we compute it on typed (i.e. legal) terms, and we can consider it as the canonical type of that term in the sense of [Kamareddine and Nederpelt 1996a].

Theorem 12(2) allows to map a term T to the environment $\gamma(T)$ obtained iterating the static type assignment on T the least number of times. Once extended arbitrarily on not well typed terms, γ yields an immersion of \mathbb{T} into \mathbb{E} . The above considerations clearly justify the choice of the function $\text{st}_g^+(E, \cdot)$ as the main ingredient for switching between terms and environments in the $\lambda\delta$ setting. Notice that γ and its properties have not being formally specified yet because the behavior of this function, especially with respect to reduction, is expected to be much clearer when the duality between terms and contexts will be achieved (see Appendix B).

3.5 Examples

If we consider the concrete sort hierarchy parameter gz defined by $gz(h) \equiv h + 1$, we have $(h_1, k_1) =_{gz} (h_2, k_2)$ iff $h_1 + k_2 = h_2 + k_1$ and we know that $\mathbb{N} \times \mathbb{N}$ (i.e. the set of the nodes) equipped with this equality is isomorphic to the set of the integer numbers. To formalize this assertion, we define the integer level equality on nodes, we extend it on compound arities, and we state the following theorem.

DEFINITION 23 INTEGER LEVEL QUALITY.

The integer level equality predicate $L_1 =_z L_2$ is defined by the rules in Figure 11.

THEOREM 13 LEVEL EQUALITY FOR THE CONCRETE PARAMETER gz .

- (1) (level equality for gz implies integer level equality)
If $L_1 =_{gz} L_2$ then $L_1 =_z L_2$.
- (2) (integer level equality implies level equality for gz)
If $L_1 =_z L_2$ then $L_1 =_{gz} L_2$.

PROOF. Both clauses are easily proved by induction on their premises. \square

The converse of Theorem 10(1) is not true in general in fact there are terms that have an arity but that are not typable. The next result shows an example.

THEOREM 14 AN UNTYPABLE TERM HAVING AN ARITY.

Given the term $T \equiv (x_2).\lambda x_3:x_0.\text{Sort}_0$ in the environment $E \equiv \lambda x_0:\text{Sort}_0.\lambda x_1:\text{Sort}_0.\lambda x_2:x_1.\text{Sort}_0$ we have that:

- (1) (*T has an arity in E*)
 $E \vdash_g T \triangleright (0, 0)$.
- (2) (*T is not typable in E*)
For all U , $E \not\vdash_g T : U$.

PROOF. Clause (1) is immediate. Clause (2) is a consequence of Theorem 7. \square

The next theorem shows that there are normal terms that do not have an arity.

THEOREM 15 A NORMAL TERM WITHOUT AN ARITY.

Given the term $T \equiv (\text{Sort}_0).\text{Sort}_0$ in the environment $E \equiv \text{Sort}_0$, we have that:

- (1) (*T is normal in E*)
 $\text{nf}(E, T)$.
- (2) (*T does not have an arity in E*)
For all L , $E \not\vdash_g T \triangleright L$.

PROOF. Both clauses are immediate consequences of simple generation lemmas, which we prove by induction on the premise with a standard technique. \square

4. THE EXTENSION OF $\lambda\delta$ WITH THE EXCLUSION BINDER χ

In this section we present the calculus $\chi\lambda\delta$ by which we mean the calculus $\lambda\delta$ extended by adding the exclusion binder χ (see Subsection 4.1). In this extension we show that every environment has a canonical well-formed form in the usual sense (see Subsection 4.2), which preserves the native type assignment.

4.1 The Calculus $\chi\lambda\delta$

In this subsection we extend $\lambda\delta$ by adding the *exclusion binder* that here we call χ (after $\chi\acute{\alpha}\omicron\sigma$: Greek for “gaping void”). The calculus we obtain is called $\chi\lambda\delta$ and is the one we formalized in [Guidi 2007a]. The idea behind the exclusion binder is that a variable x bound by χx is *excluded* in the sense that it must not occur in the scope of χx . The intended use of this binder is to replace the other binders of an environment when they are not referenced. In this way we erase these binders from the environment without changing its length. This binder-erasing technique is particularly efficient when the bound variables are referenced by position (i.e using the so-called *de Bruijn indexes* [de Bruijn 1994b]) instead of by name.

DEFINITION 24 EXCLUSION ITEM.

We introduce the syntactic item χx (exclusion) and we extend the syntax of terms and environments as follows:

$$\mathbb{T} \equiv \mathbb{T} \mid \chi\mathbb{V}.\mathbb{T} \quad \mathbb{E} \equiv \mathbb{E} \mid \chi\mathbb{V}.\mathbb{E} \quad (12)$$

scheme	redex	reductum
ζ -contraction	$\chi x.T \rightarrow_{\zeta} T$	if $x \notin \text{FV}(T)$
v -swap	$(V).\chi x.T \rightarrow_v \chi x.(V).T$	

Fig. 12. Environment-free reduction steps for exclusion

$$\frac{T_1 \Rightarrow T_2}{\chi x.T_1 \Rightarrow \chi x.T_2} \text{comp} \quad \frac{T_1 \Rightarrow T_2 \quad x \notin \text{FV}(T_1)}{\chi x.T_1 \Rightarrow T_2} \zeta \quad \frac{V_1 \Rightarrow V_2 \quad T_1 \Rightarrow T_2}{(V_1).\chi x.T_1 \Rightarrow \chi x.(V_2).T_2} v$$

Fig. 13. Reduction rules for the exclusion binder

$$\frac{C.\chi x \vdash_g T : U}{C \vdash_g \chi x.T : \chi x.U} \text{void} \quad \frac{\text{st}_g(C.\chi x, T) = U}{\text{st}_g(C, \chi x.T) = \chi x.U} \text{void} \quad \frac{C.\chi x \vdash_g T \triangleright L}{C \vdash_g \chi x.T \triangleright L} \text{void}$$

Fig. 14. Typing rules for the exclusion binder

The construction $\chi x.T$ (χ -abstraction) is thought as well formed if $x \notin \text{FV}(T)$.

We want the χ binder to have the reductional behavior of the unreferenced abbreviation, so we add the ζ -contraction and the v -swap of Figure 12.

Formally we obtain this behavior by adding the rules of Figure 13.

The general type assignment policy of the χ -abstraction follows that of the abbreviation but we do not add a rule for typing an excluded variable occurrence. In this way we capture our intuition of the exclusion because the excluded variable occurrences remain untyped. This policy applies uniformly to the assignment of the native type, of the static type and of the arity as we see in Figure 14.

The domain-based preorder on environments is extended by defining the domain of an excluded variable occurrence x as the whole set \mathbb{T} of terms because being never well formed, x can be a placeholder for any term.

DEFINITION 25 PREORDERS ON ENVIRONMENTS FOR EXCLUSION.

Under the assumption $E_2 \preceq_g E_1$ we set $E_2.\chi x \preceq_g E_1.\chi x$ and $E_2.\lambda x:W \preceq_g E_1.\lambda x$ and $E_2.\delta x \leftarrow V \preceq_g E_1.\delta x$. We do the same for the preorders \sqsubseteq_g and $\sqsubseteq_g^{\text{rc}}$.

We also need the rules stating the compatibility of the χ -abstraction with the context predicate (Definition 3), with the substitution (Definition 4, Definition 5) and with the weak reduction of environments (Definition 8).

Every theorem we stated $\lambda\delta$ holds in $\chi\lambda\delta$ as well, in addition we can prove:

THEOREM 16 MAIN PROPERTIES OF EXCLUSION.

- (1) (compatibility with environment-dependent parallel conversion)
If $C.\chi x \vdash T_1 \Leftrightarrow^* T_2$ then $C \vdash \chi x.T_1 \Leftrightarrow^* \chi x.T_2$.
- (2) (candidate exclusion)
If $(C.\chi x, (\bar{V}).T) \in [L_2]_g$ then $(C, (\bar{V}).\chi x.T) \in [L_2]_g$.
- (3) (generation lemma for native type assignment)
If $C \vdash_g \chi x.U_1 : T$ then there exists U_2 such that $C \vdash \chi x.U_2 \Leftrightarrow^* T$ and $C.\chi x \vdash_g U_1 : U_2$.

PROOF. Clause (1) is proved like Theorem 3(6). Clause (2) is proved like Theorem 6(6). Clause (3) is proved like Theorem 7(3). \square

4.2 Legal Environments in $\chi\lambda\delta$

In some versions of the λ -cube [Kamareddine et al. 1999] and in other type theories [Maietti and Sambin 2005], the rule for typing a variable declared in an environment (the so-called “start” rule) requires that the environment is legal (or well formed), which means that every declaration or definition in the environment is well typed. Following [Barendregt 1993], in Subsection 2.4 we showed that the explicit notion of a legal environment is not essential for defining our type judgement. However we may be interested in this notion for several reasons. For instance in the set theoretic semantics of a λ -calculus [Jacobs 1999], a term typed in an environment is denoted (approximately) by a function taking an argument for each environment entry, thus all the environment entries must be typable.

In this section we use the exclusion binder χ to define the “default legal version” of an arbitrary $\chi\lambda\delta$ -environment (that in particular can be a $\lambda\delta$ -environment), and we show that the type of a term is preserved when we “legalize” the environment.

Given an environment E , we introduce its default legal form $\text{wf}_g(E)$ (the abbreviation of “well formed” taken from [Coq development team 2007]) that is E with the non-binding entries removed and with the untypable entries replaced by χ .

By using the χ binder, the environment $\text{wf}_g(E)$ has the length of the environment E and the terms referring to E can refer to $\text{wf}_g(E)$ without being relocated. This feature is desired in the formal specification of $\chi\lambda\delta$ [Guidi 2007a], where the environment entries are referred by position, and not by name as in this paper.

Notice that the function wf_g is well defined and total because the type inference problem is decidable in $\chi\lambda\delta$ (see Theorem 11(2)). Also notice that wf_g depends on the sort hierarchy parameter g defined in Subsection 2.4.

In [Guidi 2007a] we do not have the function for inferring the type of a term, therefore we prefer to define wf_g by axiomatizing the proposition $\text{wf}_g(E_1) = E_2$.

DEFINITION 26 ENVIRONMENT LEGALIZATION.

The default legalization of the environment E is the environment $\text{wf}_g(E)$ defined by axiomatizing the predicate $\text{wf}_g(E_1) = E_2$ with the following clauses:

- (1) $\text{wf}_g(\text{Sort}_h) = \text{Sort}_h$.
- (2) If $\text{wf}_g(E_1) = E_2$ and $E_1 \vdash_g V : W$ then $\text{wf}_g(E_1.\delta x \leftarrow V) = E_2.\delta x \leftarrow V$.
- (3) If $\text{wf}_g(E_1) = E_2$ and $E_1 \vdash_g W : V$ then $\text{wf}_g(E_1.\lambda x : W) = E_2.\lambda x : W$.
- (4) If $\text{wf}_g(E_1) = E_2$ then $\text{wf}_g(E_1.\chi x) = E_2.\chi x$.
- (5) If $\text{wf}_g(E_1) = E_2$ and for each W , $E_1 \not\vdash_g V : W$, then $\text{wf}_g(E_1.\delta x \leftarrow V) = E_2.\chi x$.
- (6) If $\text{wf}_g(E_1) = E_2$ and for each V , $E_1 \not\vdash_g W : V$, then $\text{wf}_g(E_1.\lambda x : W) = E_2.\chi x$.
- (7) If $\text{wf}_g(E_1) = E_2$ then $\text{wf}_g(E_1.(V)) = E_2$.
- (8) If $\text{wf}_g(E_1) = E_2$ then $\text{wf}_g(E_1.(W)) = E_2$.

We do not give these axioms as rules because Axiom (5) and Axiom (6) are expressed in the meta-language and can not be given in rule form.

The most relevant properties of the function wf_g are listed in the theorem below:

THEOREM 17 MAIN PROPERTIES OF THE LEGALIZATION FUNCTION.

- (1) *(the legalization function is total)*
For all C_1 , there exists C_2 such that $\text{wf}_g(C_1) = C_2$.

- (2) (*preservation of the native type assignment*)
 If $C_1 \vdash_g T : U$ and $\text{wf}_g(C_1) = C_2$ then $C_2 \vdash_g T : U$.
- (3) (*environments in native type assignments can be assumed legal*)
 If $C_1 \vdash_g T : U$ then there exists C_2 such that $\text{wf}_g(C_1) = C_2$ and $C_2 \vdash_g T : U$.

PROOF. Clause (1) is proved by induction on C_1 with the help of Theorem 11(2). Clause (2) is proved by induction on its first premise; here we need Theorem 8(2), Theorem 9(2) and Theorem 11(2). Clause (3) is implied the previous clauses. \square

Theorem 17(2) and *Theorem 17(3)* imply each other but we noticed that the second one is slightly harder to prove directly because its conclusion is existential.

5. CONCLUSIONS AND FUTURE WORK

In this paper we take the calculus Λ_∞ [van Benthem Jutting 1994c] with the restricted applicability condition used by Pure Type Systems [Barendregt 1993], to which we add non-recursive untyped abbreviations, an infinite number of typed sorts, explicit type annotations, and some reduction schemes involving these constructions. Remarkably we also replace the call-by-value β -contraction scheme with its call-by-value version. Then we show that the resulting typed λ -calculus, that we term $\lambda\delta$, satisfies some important desirable properties such as the confluence of reduction, the correctness of types, the uniqueness of types up to conversion, the subject reduction of the type assignment, the strong normalization of the typed terms and, as a corollary, the decidability of type inference problem.

$\lambda\delta$ features the unification of terms and types, the immersion of environments into terms, a “compatible” typing policy in which the dynamic aspect of the type assignment is confined in the “conversion rule” and finally a predicative abstraction.

The author conjectures that the expressive power of $\lambda\delta$ is that of λP .

We see an application of this calculus as a formal specification language for the type theories, like mTT [Maietti and Sambin 2005] or CTT [Nordström et al. 1990; Martin-Löf 1984], that require to be expressed in a predicative foundation. In this sense $\lambda\delta$ can be related both to PAL⁺ [Luo 2003] and to Martin-Löf’s theory of expressions [Nordström et al. 1990], that pursue the same aim and use the type system of $\lambda \rightarrow$ (i.e. they use arities). Namely the author conjectures that $\lambda\delta$ includes both these theories. In particular these calculi use k -uples of terms and $\lambda\delta$ can provide for this construction as well (see Appendix B.2).

The advantage of $\lambda\delta$ on these calculi is that the structural rules of mTT and CTT can be justified by the rules of our calculus (see Appendix A).

As an additional feature, the extension of $\lambda\delta$ termed $\chi\lambda\delta$ (Subsection 4.1) comes with a full machine-checked specification of its properties (see Subsection 1.3).

In this section we will discuss some design features of $\chi\lambda\delta$ (Subsection 5.1) and we will summarize the open issues of the calculus (Subsection 5.2).

5.1 The Block Structure of $\chi\lambda\delta$

$\chi\lambda\delta$ was carefully designed by the author on the basis of the criteria discussed in Subsection 1.2. Another important design issue of this calculus is its *block structure*, where by a *block* we mean a subset of constructions and reduction rules tightly connected to each other that we see as a unit (see Figure 15).

block id	main item	main item denomination
8	χx	unconditioned exclusion
5	$\lambda x:W$	abstraction over a complete type W
1	$\delta x \leftarrow V$	unconditioned abbreviation of V
0	Sort_h	sort of level h
-1	x	variable occurrence

Fig. 15. Block hierarchy in $\chi\lambda\delta$

item	domain	applicator with \rightarrow_v	reduction	\rightarrow_ζ	annotator with \rightarrow_τ
χx	$\{T \mid \top\}$	no	no	yes	no
$\lambda x:V$	$\{T \mid E \vdash_g T : V\}$	(V)	\rightarrow_β	no	$\langle W \rangle$
$\delta x \leftarrow V$	$\{T \mid T = V\}$	no	\rightarrow_δ	yes	no

Fig. 16. Detailed structure of the blocks about binding items

$\chi\lambda\delta$ has one block for each non-recursive construction and one for each binder.

The author assigned a numeric identifier to each block just to suggest a hierarchy in the block structure. The type W on which we abstract using $\lambda x:W$ is *complete* because it represents a complete specification of the functional structure of its inhabitants (see the comments on Figure 7(abst)). The abbreviation introduced by $\delta x \leftarrow V$ is *unconditioned* because it can always be unfolded by reduction.

Generally speaking each binder has a *domain* by which we mean the class of the terms that can be substituted for the variable occurrences referring to that binder. Moreover a binder is here called *conditioned* if it has an applicator item associated to a specific reduction rule. The applicator item always swaps with a binder of a different block by means of a v reduction step (see Figure 2 and Figure 13) and the specific reduction rule always contracts the applicator-binder pair to an unconditioned abbreviation. An unconditioned binder is always eliminable by reduction when it is not an environment entry. if this domain is specified up to a non-trivial equivalence relation, its inhabitants can be annotated with a preferred specification of this domain. The annotator item can always be removed by reduction.

These considerations are summarized in Figure 16 where the λ -abstraction is considered in an environment E . Notice that the abbreviation and the exclusion do not have an applicator with a specific reduction because they are unconditioned.

5.2 Open Issues

As already stressed along the paper, our presentation of $\lambda\delta$ leaves some open issues that we want to reconsider in this subsection.

First of all, some technical aspects of the calculus need to be improved: this includes taking a final decision on the shape of Definition 18 and of Definition 9.

In particular we plan to reformulate the reduction predicates without the explicit substitution (Definition 4, Definition 5, Definition 6). and we want to reformulate the arity assignment without the level equality (Definition 17) that is undecidable in general. We might also want to add the following type assignment rule:

$$\frac{E \vdash_g T : U \quad E \vdash_g (V).U : W}{E \vdash_g (V).T : (V).U} \text{ appl2} \quad (13)$$

with which we expect to type in $\lambda\delta$ all terms typable in Λ_∞ (see Subsection 1.2).

The items $\lambda y:D$, $\delta y \leftarrow F$, (F) and $\langle D \rangle$ are not allowed at the moment (recall that D and F stand for environments), but when $\lambda\delta$ will be extended by considering them as well, a duality between terms and environments will arise (see Appendix B).

Secondly there are some conjectures that need to be proved formally. In particular we are interested in understanding if the problem of type inhabitation is decidable (this is an important property of $\lambda \rightarrow$, see [Barendregt 1993]).

Thirdly we might want to extend $\chi\lambda\delta$ adding more blocks in the sense of Subsection 5.1. Namely there are five constructions that can be of interest: declared constants (block 4), meta-variables (block -2), parameters, (block 7), conditioned abbreviations (block 3) and abstractions over incomplete types (block 6).

The first three constructions are taken from real implementations of typed λ -calculus. In particular we see the declaration of a constant as the unconditioned version of the λ -abstraction, which we would like to denote with $\lambda_o x:V$ (where the o can mean *opaque* or can be an *omicron* chosen after $\acute{o}\nu\omicron\mu\alpha$: Greek for “name”).

Parameters appear in many logical frameworks [Kamareddine et al. 2004; Luo 2003]. Conditioned abbreviations are based on the binder $\delta_c x \leftarrow V$, on the applicator $(V)_c$ and on the reduction rule $(V)_c.\delta_c x \leftarrow V.T \rightarrow_{\beta_c} \delta x \leftarrow V.T$. They provide for possibly unexpandable abbreviations and mainly the applicator $(V)_c$ does not carry any information into a β -redex except for its presence (since the term V appears in the binder). So we suspect that $(V)_c$ can be related to a *connection* of a Whole Adaptive System [Solmi 2005] and we call $(V)_c$ a *connessionistic* application item.⁸

Abstractions over incomplete types (i.e. types that do not specify the functional structure of their inhabitants completely) are meant to simulate the Π -abstractions of the λ -cube [Barendregt 1993] and the author sees fitting the Π binder into the architecture of $\lambda\delta$ as a very challenging task. In particular it would be interesting to relate this extension of $\lambda\delta$ to COC since this calculus has been fully specified in COQ [Barras 1996] as well as $\lambda\delta$ itself, and the author sees the possibility of certifying rigorously the mappings that may exist between these systems.

The novelty of $\lambda\delta$ extended with Π would be that Π could appear at the level of terms and inside environments rather than only at the level of types.

In the perspective of relating this extension with a COC with universes, we would also need a mechanism that makes Sort_h a sub-sort of Sort_k when $h < k$.

A. JUSTIFYING THE STRUCTURAL FRAGMENT OF MTT WITH $\lambda\delta$

In the present appendix we show how the structural rules of Minimal Type Theory (mTT) [Maietti and Sambin 2005] can be justified through the rules of $\lambda\delta$ and we proceed in three steps. In Appendix A.1 we show that $\lambda\delta$ can be used as a theory of expressions for mTT. In Appendix A.2 we show that $\lambda\delta$ type assignment and conversion judgements can model mTT judgements. In Appendix A.3 we show that $\lambda\delta$ rules can model mTT structural rules. In order to achieve this objective, we propose to remove η -conversion and the so-called Cont judgement from mTT, and to perform some changes to the mTT rules called *var* and *prop-into-set*.

⁸Describing the computational model of a Whole Adaptive System in terms of a typed λ -calculus requires much more than conditioned abbreviations: in particular we feel that anti-binders, in the sense of [Hendriks and van Oostrom 2003], might play an important role for this task.

Our justification is based on a straight forward mapping of judgements, which exploits uniformly dependent types on the $\lambda\delta$ side. The underlying idea is to map the inhabitation judgements to the type judgement \vdash : (at different levels of the type hierarchy) and the equality judgements to the conversion judgement $\vdash \Leftrightarrow^*$.

When referring to mTT we will use the notation of [Maietti and Sambin 2005].

A.1 $\lambda\delta$ can serve as a Theory of Expressions for mTT

According to [Maietti and Sambin 2005] the theory of expressions underlying mTT is the one, originally due to Martin-Löf, underlying CTT [Nordström et al. 1990] without combinations and selections. Moreover typed abstractions (à la Church) are used in place of untyped ones.

Therefore mTT-expressions are based on variables, primitive constants, defined constants, applications and typed abstractions.

Moreover every meaningful mTT-expression has an *arity*, which is a type expression of the instance of $\lambda \rightarrow$ with one type constant 0.

Equality between mTT-expressions is defined up to *definitional equality*: a rewriting mechanism that incorporates $\alpha\beta\eta$ -conversion, and δ -conversion (equality between the definiendum and the definiens of an abbreviation).

In our proposal we leave η -conversion aside because we suspect that this conversion is not strictly necessary in mTT and is used just as syntactic sugar. In any case η -conversion is available for λ -abstractions as expected (see Theorem 3(8)).

As a matter of fact $\lambda\delta$ can handle the mentioned ingredients as follows.

Variables, defined constants, applications and typed abstractions are term constructions of the calculus (see Definition 1). In particular we regard all definitions as δ -entries of a *global environment* E_y in which we close every term.

Primitive constants are regarded as references to λ -entries (i.e. declarations) of the environment E_y . So E_y contains declarations and definitions.

Types can be substituted for arities. Notice that arities exist in $\lambda\delta$ as well (see Definition 18) and that typed terms have an arity (see Theorem 10(1)).

Finally **definitional equality** is handled through environment-dependent parallel conversion (see Definition 9) that incorporates $\alpha\beta\delta$ -conversion.

A.2 $\lambda\delta$ Judgements can express mTT Judgements

mTT features six main judgements that fall into two classes: declarations and equalities. Declarations state that an expression is a legal proposition, a legal data type, or a legal element of a data type. Equalities state that two legal propositions, data types, or elements of a data type are semantically equal.

Parametric expressions are allowed and each main judgement includes an *explicit environment* where the local parameters are declared.

Other parameters, shared among all judgements of a given rule, are declared in an *implicit environment* extracted from the premises of that rule.

Summing up, a legal mTT-expression requires three environments: the explicit environment (provided by the judgement containing that expression), the implicit environment (extracted from the premises of the rule containing that judgement) and the global environment (for global declarations and abbreviations).

A judgement stating that an explicit environment is legal, is also provided.

We can map these judgements to $\lambda\delta$ -judgements in the way we explain below.

Sort hierarchy. We need two sorts Prop and Set that we regard as aliases of Sort_0 and Sort_1 respectively (we can include these abbreviations in the global environment C_y). We also set the sort hierarchy parameter (see Subsection 2.4) to the function $g2z$ such that $g2z(h) \equiv h + 2$. This is the simplest choice ensuring that the positions of Set and Prop in the sort hierarchy graph (see Subsection 2.6) are disconnected. In particular we observe that if $gz(h) \equiv h + 1$ (as in Subsection 3.5) we derive directly from Figure 7(sort): $E \vdash_{gz} \text{Set} : \text{Prop}$, which is against the intuition.

Environments. The explicit environment of an mTT-judgement has the form: $\Gamma \equiv x_1 \in A_1 \text{ Set}, \dots, x_n \in A_n \text{ Set}$ where x_i is a variable and A_i is an expression.

We can map each declaration of Γ in a λ -entry, so Γ itself becomes the environment $C_x \equiv \lambda x_1:A_1 \dots \lambda x_n:A_n.\text{Set}$ of $\lambda\delta$.

The implicit environment of an mTT-judgement does not need an explicit mapping since we can exploit the implicit environment of the corresponding judgement of $\lambda\delta$ (at least as long as we are dealing just with the structural rules of mTT).

Declarations: $A \text{ Prop } [\Gamma], A \text{ Set } [\Gamma], a \in A \text{ Set } [\Gamma], \Gamma \text{ Cont.}$

A declaration judgement is mapped to a type assignment judgement (see Definition 13). Namely we map $A \text{ Prop } [\Gamma]$ to $C_y.C_x \vdash_{g2z} A : \text{Prop}$, we map $A \text{ Set } [\Gamma]$ to $C_y.C_x \vdash_{g2z} A : \text{Set}$ and we map $a \in A \text{ Set } [\Gamma]$ to $C_y.C_x \vdash_{g2z} a : A$ in the implicit environment $C_y.C_x \vdash_{g2z} A : \text{Set}$. Here $C_y.C_x$ refers to the concatenation of C_y and C_x . Notice that the type assignment is invariant for conversion (modelling definitional equality) as stated by Figure 7(conv) and Theorem 9(6).

Coming to the legal explicit environment judgement $\Gamma \text{ Cont}$, the experience of the author with $\lambda\delta$ shows that such a judgement is useless (as it does not guarantee additional meta-theoretical properties) and heavy (as it introduces a mutual dependence between itself and $A \text{ Set } [\Gamma]$ at the meta-theory level). The point is that an unreferenced parameter does not need a legal declaration unless it is the formal argument of a function. So we propose not to map $\Gamma \text{ Cont}$ and to change the related rules (see Appendix A.3). In any case legal environments are supported in the calculus $\chi\lambda\delta$ (Subsection 4.2) if they are needed for some reason.

Equalities: $A_1 = A_2 \text{ Prop } [\Gamma], A_1 = A_2 \text{ Set } [\Gamma], a_1 = a_2 \in A \text{ Set } [\Gamma]$.

An equality judgement is mapped to an environment-dependent conversion judgement (see Definition 9). Namely, we map $A_1 = A_2 \text{ S } [\Gamma]$ to $C_y.C_x \vdash A_1 \Leftrightarrow^* A_2$ in the implicit environment $C_y.C_x \vdash_{g2z} A_1 : S$ and $C_y.C_x \vdash_{g2z} A_2 : S$ where S is either Prop or Set, and we map $a_1 = a_2 \in A \text{ Set } [\Gamma]$ to $C_y.C_x \vdash a_1 \Leftrightarrow^* a_2$ in the implicit environment $C_y.C_x \vdash_{g2z} a_1 : A$, $C_y.C_x \vdash_{g2z} a_2 : A$ and $C_y.C_x \vdash_{g2z} A : \text{Set}$.

Notice that the conversion judgement is invariant for conversion itself (modelling definitional equality) because the conversion is an equivalence relation.

A.3 $\lambda\delta$ Rules can express mTT Structural Rules

Our proposal for the structural rules of mTT is shown in Figure 17.

the prop-into-set rule can not be modelled, as it is, by $\lambda\delta$ because $\lambda\delta$ does not feature subtyping. Therefore our proposal is to make the coercion from Prop to Set explicit. Namely we declare a primitive constant pr of type $\lambda x:\text{Prop}.\text{Set}$ in the global environment C_y and we set Figure 17(ps) modelled by Figure 7(appl). This solution is well known in the literature (see [Coquand and Huet 1988; van Benthem Jutting 1994b; de Bruijn 1994c]).

$$\begin{array}{c}
\frac{A \text{ Prop } [\Gamma]}{pr(A) \text{ Set } [\Gamma]} \text{ ps} \quad \frac{A \text{ Set } [\Gamma]}{x \in A \text{ Set } [\Gamma, x \in A \text{ Set}, \Delta]} \text{ var} \\
\frac{a \in A_1 \text{ Set } [\Gamma] \quad A_1 = A_2 \text{ Set } [\Gamma]}{a \in A_2 \text{ Set } [\Gamma]} \text{ seteq} \\
\frac{A \text{ Set } [\Gamma]}{A = A \text{ Set } [\Gamma]} \text{ r} \quad \frac{A_1 = A_2 \text{ Set } [\Gamma]}{A_2 = A_1 \text{ Set } [\Gamma]} \text{ s} \quad \frac{A_1 = A \text{ Set } [\Gamma] \quad A = A_2 \text{ Set } [\Gamma]}{A_1 = A_2 \text{ Set } [\Gamma]} \text{ t} \\
\frac{A \text{ Prop } [\Gamma]}{A = A \text{ Prop } [\Gamma]} \text{ r} \quad \frac{A_1 = A_2 \text{ Prop } [\Gamma]}{A_2 = A_1 \text{ Prop } [\Gamma]} \text{ s} \quad \frac{A_1 = A \text{ Prop } [\Gamma] \quad A = A_2 \text{ Prop } [\Gamma]}{A_1 = A_2 \text{ Prop } [\Gamma]} \text{ t} \\
\frac{a \in A \text{ Set } [\Gamma]}{a = a \in A \text{ Set } [\Gamma]} \text{ r} \quad \frac{a_1 = a_2 \in A \text{ Set } [\Gamma]}{a_2 = a_1 \in A \text{ Set } [\Gamma]} \text{ s} \quad \frac{a_1 = a \in A \text{ Set } [\Gamma] \quad a = a_2 \in A \text{ Set } [\Gamma]}{a_1 = a_2 \in A \text{ Set } [\Gamma]} \text{ t} \\
\frac{A \text{ Set } [\Gamma] \quad B \text{ Set } [\Gamma, x \in A \text{ Set}]}{(x : A)B (x : A) \text{ Set } [\Gamma]} \text{ i} \quad \frac{A \text{ Set } [\Gamma] \quad B_1 = B_2 \text{ Set } [\Gamma, x \in A \text{ Set}]}{(x : A)B_1 = (x : A)B_2 (x : A) \text{ Set } [\Gamma]} \text{ i} \\
\frac{A \text{ Set } [\Gamma] \quad B \text{ Prop } [\Gamma, x \in A \text{ Set}]}{(x : A)B (x : A) \text{ Prop } [\Gamma]} \text{ i} \quad \frac{A \text{ Set } [\Gamma] \quad B_1 = B_2 \text{ Prop } [\Gamma, x \in A \text{ Set}]}{(x : A)B_1 = (x : A)B_2 (x : A) \text{ Prop } [\Gamma]} \text{ i} \\
\frac{A \text{ Set } [\Gamma] \quad b \in B \text{ Set } [\Gamma, x \in A \text{ Set}]}{(x : A)b \in (x : A)B (x : A) \text{ Set } [\Gamma]} \text{ i} \quad \frac{A \text{ Set } [\Gamma] \quad b_1 = b_2 \in B \text{ Set } [\Gamma, x \in A \text{ Set}]}{(x : A)b_1 = (x : A)b_2 \in (x : A)B (x : A) \text{ Set } [\Gamma]} \text{ i} \\
\frac{a \in A \text{ Set } [\Gamma] \quad B (x : A) \text{ Set } [\Gamma]}{B(a) \text{ Set } [\Gamma]} \text{ e} \quad \frac{a \in A \text{ Set } [\Gamma] \quad B_1 = B_2 (x : A) \text{ Set } [\Gamma]}{B_1(a) = B_2(a) \text{ Set } [\Gamma]} \text{ e} \\
\frac{a \in A \text{ Set } [\Gamma] \quad B (x : A) \text{ Prop } [\Gamma]}{B(a) \text{ Prop } [\Gamma]} \text{ e} \quad \frac{a \in A \text{ Set } [\Gamma] \quad B_1 = B_2 (x : A) \text{ Prop } [\Gamma]}{B_1(a) = B_2(a) \text{ Prop } [\Gamma]} \text{ e} \\
\frac{a \in A \text{ Set } [\Gamma] \quad b \in (x : A)B (x : A) \text{ Set } [\Gamma]}{b(a) \in (x : A)B(a) \text{ Set } [\Gamma]} \text{ e} \\
\frac{a \in A \text{ Set } [\Gamma] \quad b_1 = b_2 \in (x : A)B (x : A) \text{ Set } [\Gamma]}{b_1(a) = b_2(a) \in (x : A)B(a) \text{ Set } [\Gamma]} \text{ e}
\end{array}$$

Fig. 17. Our proposal for the structural rules of mTT

The *var* rule. Our proposal for this rule is Figure 17(var) modelled by Figure 7(decl). The implicit environment is respected because of Theorem 8(1).

The *seteq* rule. This rule is Figure 17(seteq) modelled by Figure 7(conv) whose first premise is taken from the implicit environment.

The equivalence rules of the equality judgements are justified by the fact that the environment-dependent conversion is an equivalence relation.

The complete list is in Figure 17 (labels: r, s, t).

The derivable rules. Notice that [Nordström et al. 1990] suggests some additional structural rules (like a second seteq rule and some substitution rules) that are not included in mTT because they are derivable. In the $\lambda\delta$ perspective we derive these rules from Theorem 3(4), Theorem 3(5), Theorem 8(5) and Theorem 8(7).

The rules on classes. If we regard Prop and Set as primitive constants rather than judgement keywords, we can build expressions like $(x_1 : e_1) \dots (x_n : e_n) \text{Set}$ or $(x_1 : e_1) \dots (x_n : e_n) \text{Prop}$ (called *types* in mTT or *categories* in CTT [Martin-Löf 1984]). With these “classes” we can form the following judgements:

$$\begin{array}{ll}
B (x : A) \text{Set } [\Gamma] & B_1 = B_2 (x : A) \text{Set } [\Gamma] \\
B (x : A) \text{Prop } [\Gamma] & B_1 = B_2 (x : A) \text{Prop } [\Gamma] \\
b \in B (x : A) \text{Set } [\Gamma] & b_1 = b_2 \in B (x : A) \text{Set } [\Gamma]
\end{array} \quad (14)$$

that we explain with the rules modelled by Figure 7(abst) and Theorem 3(6). These rules are shown in Figure 17 with the label: i. The elimination rules, modelled by Figure 7(appl) and Theorem 3(4), are shown in Figure 17 with the label: e.

B. TOWARDS A DUALITY BETWEEN TERMS AND ENVIRONMENTS

The present appendix contains some hints on how the author plans to complete $\lambda\delta$ by adding the items $\lambda y:D, \delta y \leftarrow F, (F)$ and $\langle D \rangle$ both in the terms and in the environments. In principle the need for these items was evident from the very start but they were not included in [Guidi 2007a] because of the technical problems they seemed to give. In particular the author did not see the importance of the iterated static type assignment as a way to map \mathbb{T} into \mathbb{E} (Subsection 2.5) until the properties of $\lambda\delta$ were made clear (especially Theorem 12(2), Theorem 12(1) and Theorem 10(1)). We would like to stress that the contents of this appendix are just a proposal for future research on $\lambda\delta$ and have not been certified yet.

In Appendix B.1 we introduce these new items, In Appendix B.2 we propose the new term construction $\{F\}.T$ as an application, in Appendix B.3 we propose to merge \mathbb{T} and \mathbb{E} in a single data type to avoid the replication of dual definitions and theorems in the perspective of certifying the properties of complete $\lambda\delta$.

B.1 Complete $\lambda\delta$: Dualizing Terms and Environments

According to Definition 1 the argument of the abstractors, abbreviators, applicators and type annotators is a term. Nevertheless an environment can be allowed as well.

DEFINITION 27 COMPLETE SYNTAX OF TERMS AND ENVIRONMENTS.

The complete versions of \mathbb{T} and \mathbb{E} are defined by extending Definition 1 as follows:

$$\mathbb{T} \equiv \mathbb{T} \mid \lambda W:\mathbb{E}.T \mid \delta W \leftarrow \mathbb{E}.T \mid (\mathbb{E}).T \mid \langle \mathbb{E} \rangle.T \quad (15)$$

$$\mathbb{E} \equiv \mathbb{E} \mid W \mid \lambda W:\mathbb{E}.\mathbb{E} \mid \delta W \leftarrow \mathbb{E}.\mathbb{E} \mid (\mathbb{E}).\mathbb{E} \mid \langle \mathbb{E} \rangle.\mathbb{E} \quad (16)$$

where W is a set of names for variables denoting environments.

We call a recursive construction positive when its arguments belong to the same type and negative otherwise. We call this attribute the polarity of the construction.

Notice that the calculi of the $\lambda\mu$ family use two different sets of variables as well.

Once defined in this way, \mathbb{T} and \mathbb{E} are isomorphic through the polarity preserving transformations $\mathcal{E} : \mathbb{T} \rightarrow \mathbb{E}$ and $\mathcal{T} : \mathbb{E} \rightarrow \mathbb{T}$ defined below.

DEFINITION 28 THE TRANSFORMATIONS \mathcal{E} AND \mathcal{T} .

The transformations $\mathcal{E} : \mathbb{T} \rightarrow \mathbb{E}$ and $\mathcal{T} : \mathbb{E} \rightarrow \mathbb{T}$ work as follows:

- (1) $\mathcal{E}[\text{Sort}_h] = \text{Sort}_h$ and $\mathcal{T}[\text{Sort}_h] = \text{Sort}_h$;
- (2) $\mathcal{E}[x] = y$ and $\mathcal{T}[y] = x$ (here we assume that \mathbb{V} and \mathbb{W} are isomorphic);
- (3) $\mathcal{E}[\lambda x:W.T] = \lambda y:\mathcal{E}[W].\mathcal{E}[T]$ and $\mathcal{T}[\lambda y:D.E] = \lambda x:\mathcal{T}[D].\mathcal{T}[E]$;
- (4) $\mathcal{E}[\lambda y:D.T] = \lambda x:\mathcal{T}[D].\mathcal{E}[T]$ and $\mathcal{T}[\lambda x:W.E] = \lambda y:\mathcal{E}[W].\mathcal{T}[E]$;
- (5) $\mathcal{E}[\delta x \leftarrow V.T] = \delta y \leftarrow \mathcal{E}[V].\mathcal{E}[T]$ and $\mathcal{T}[\delta y \leftarrow F.E] = \delta x \leftarrow \mathcal{T}[F].\mathcal{T}[E]$;
- (6) $\mathcal{E}[\delta y \leftarrow F.T] = \delta x \leftarrow \mathcal{T}[F].\mathcal{E}[T]$ and $\mathcal{T}[\delta x \leftarrow V.E] = \delta y \leftarrow \mathcal{E}[V].\mathcal{T}[E]$;
- (7) $\mathcal{E}[(V).T] = (\mathcal{E}[V]).\mathcal{E}[T]$ and $\mathcal{T}[(F).E] = (\mathcal{T}[F]).\mathcal{T}[E]$;
- (8) $\mathcal{E}[(F).T] = (\mathcal{T}[F]).\mathcal{E}[T]$ and $\mathcal{T}[(V).E] = (\mathcal{E}[V]).\mathcal{T}[E]$;
- (9) $\mathcal{E}[\langle W \rangle.T] = \langle \mathcal{E}[W] \rangle.\mathcal{E}[T]$ and $\mathcal{T}[\langle D \rangle.E] = \langle \mathcal{T}[D] \rangle.\mathcal{T}[E]$;
- (10) $\mathcal{E}[\langle D \rangle.T] = \langle \mathcal{T}[D] \rangle.\mathcal{E}[T]$ and $\mathcal{T}[\langle W \rangle.E] = \langle \mathcal{E}[W] \rangle.\mathcal{T}[E]$.

Definition 27 opens some issues: we discuss the most relevant below.

Focalized terms. When a term reference x points to an abstractor $\lambda x:W$ in an environment E it may be the case that the rightmost item of E is not a sort. In that event we must consider its iterated static type (see Theorem 12(2)). More precisely if E is $C.y$ and if y points to $\lambda y:D$ or to $\delta y\leftarrow F$, we recursively resolve x in the environments $C.D$ or $C.F$ respectively (this is much like considering the iterated static type of E except for the rightmost sort item that is irrelevant when searching for binders). This solution may look strange at a first glance but consider $E = \lambda y:D.y$: this is the empty environment whose “hole” is y in the sense of [Curien and Herbelin 2000]. Normally the references to the empty environment are not legal but in our case the “hole” is typed explicitly so we can foresee its contents by inspecting its type. This means that for $D = \lambda x:W.\text{Sort}_{g(n)}$ the focalized term $(\lambda y:D.y, x)$ is legal and the term reference x points to $\lambda x:W$. Furthermore that reference continues to point to the same binder when E is instantiated and reduced:

- (1) Legal instantiation with $F = \lambda x:W.\text{Sort}_n$: $((F).\lambda y:D.y, x)$.
- (2) β -contraction: $(\delta y\leftarrow F.y, x)$.
- (3) δ -expansion: $(\delta y\leftarrow F.F, x)$.
- (4) ζ -contraction: (F, x) .

As we see, everything works fine because the item $\lambda x:W$ must appear in F as well as in D in order for the instantiation to be legal (i.e. well typed).

If the term reference x points to an abbreviator $\delta x\leftarrow V$, we do the same thing.

Pushing. When moving an abstractor $\lambda x:W$ from a term to an environment, as we might need to do when the term and the environment themselves are the components of a focalized term, we must make sure that the references to $\lambda x:W$ are preserved. So, when the environment has the form $C.y$ where y points to $\lambda y:D$ or to $\delta y\leftarrow F$, we must move $\lambda x:W$ recursively into D or F respectively. In the first case this amounts to updating the explicit type D of the environment “hole” in a way that makes it possible to fill that “hole” through a legal instantiation.

As before, when we move an abbreviator $\delta x\leftarrow V$, we do the same thing.

Reduction. The β -redexes are $(V).\lambda x:W$ (from Subsection 2.3) and symmetrically $(F).\lambda y:D$. The abbreviations $\delta x\leftarrow V.E$ do not ζ -reduce (from [Guidi 2006]) and symmetrically the abbreviations $\delta y\leftarrow F.T$ do not ζ -reduce either.

B.2 Environments as Aggregates

Formally the k -uple (V_{k-1}, \dots, V_0) at position $(h, 0)$ in the type hierarchy is denoted by the environment $E = \delta x_{k-1}\leftarrow V_{k-1} \dots \delta x_0\leftarrow V_0.\text{Sort}_h$.

More generally the binders $\lambda x:W$ and $\delta x\leftarrow V$ of an environment E (as well as the binders $\lambda y:D$ and $\delta y\leftarrow F$ of a term T) can be seen as the fields of an aggregate structure. These fields can be definitions (denoted by the $\delta x\leftarrow V$ items) or declarations (denoted by the $\lambda x:W$ items) and can be dependent. In order to be effective, aggregates need a projection mechanism that allows to read their fields. To this aim we propose the item $\{F\}$ that we call *projector* and the term constructions $\{F\}.T$ that we call *projection*. Considering the previous k -uple E , the basic idea is that $\{E\}.x_i$ must reduce to V_i , so we set the following sequential reduction rule.

$$\text{If } F \vdash T_1 \rightarrow T_2 \text{ and if } T_2 \text{ does not refer to } F \text{ then } \{F\}.T_1 \rightarrow_\pi T_2 \quad (17)$$

Notice that $\{F\}.T$ might be related to the *with* instruction of the PASCAL programming language [Jensen and Wirth 1981] and might look like: *with F do T*.

Following the “environments as aggregates” interpretation, we might expect to type E with $C_1 = \lambda x_{k-1}:W_{k-1} \dots \lambda x_0:W_0.\text{Sort}_{g(h)}$ where each W_i is the type of V_i . Nevertheless the type of E as a term is $C_2 = \delta x_{k-1} \leftarrow V_{k-1} \dots \delta x_0 \leftarrow V_0.\text{Sort}_{g(h)}$ according to Definition 13 but notice that $C_2 \preceq_g C_1$ (this is the domain-based preorder of Subsection 2.7). This consideration shows that it could make sense to investigate the extension of $\lambda\delta$ with a subtyping relation based on \preceq_g .

B.3 Unified $\lambda\delta$: Introducing Polarized Terms

In this subsection we propose the notion of a *polarized term*: an expression capable of representing both a term and an environment (in the sense of Definition 27) in a way that turns the transformations \mathcal{E} and \mathcal{T} into the identity functions.

The basic idea consists in decorating the recursive term constructions with the information on their polarity represented as a boolean value.

Let us denote the data type of the boolean values with $2 \equiv \{-, +\}$ and let us assume that $+$ (positive polarity) represents \top , then a polarized term is as follows.

DEFINITION 29 SYNTAX OF POLARIZED TERMS.

The set of polarized terms is defined as follows:

$$\mathbb{P} \equiv \text{Sort}_{\mathbb{N}} \mid \forall \mid 2\lambda\forall:\mathbb{P}.\mathbb{P} \mid 2\delta\forall\leftarrow\mathbb{P}.\mathbb{P} \mid 2(\mathbb{P}).\mathbb{P} \mid 2\langle\mathbb{P}\rangle.\mathbb{P} \quad (18)$$

Definition 29 opens the issue of deciding whether a $Q \in \mathbb{P}$ can be mapped back to a $V \in \mathbb{T}$ or to an $F \in \mathbb{E}$. Clearly the fact that the transformations \mathcal{E} and \mathcal{T} are mapped to the identity functions on \mathbb{P} says that this information, which we call the *absolute polarity* of Q , is not recoverable. What we can recover is the *relative polarity* of Q with respect to a superterm P of Q . This is to say that we can know if P and Q represent two elements of the same type or not.

DEFINITION 30 RELATIVE POLARITY ASSIGNMENT.

The partial function $\text{polarity}[P, Q]$, that returns $+$ if the terms P and Q have the same absolute polarity, is defined by the clauses shown below, where \leftrightarrow denotes the boolean coimplication (i.e. the negated exclusive disjunction).

- (1) (*refl*) $\text{polarity}[P, P] = +$;
- (2) (*trans*) if $\text{polarity}[P_1, P] = b_1$ and $\text{polarity}[P, P_2] = b_2$ then $\text{polarity}[P_1, P_2] = b_1 \leftrightarrow b_2$;
- (3) (*fst*) $\text{polarity}[b\lambda z:Q.P, P] = +$; $\text{polarity}[b\lambda z:Q.P, P] = +$;
 $\text{polarity}[b(Q).P, P] = +$; $\text{polarity}[b\langle Q \rangle.P, P] = +$;
- (4) (*snd*) $\text{polarity}[b\lambda z:Q.P, Q] = b$; $\text{polarity}[b\delta z\leftarrow Q.P, Q] = b$;
 $\text{polarity}[b(Q).P, Q] = b$; $\text{polarity}[b\langle Q \rangle.P, Q] = b$.

We conjecture that the knowledge of relative polarity is enough to treat the version of $\lambda\delta$ based on polarized terms. We call this calculus *unified $\lambda\delta$* or $1\lambda\delta$.

As an example let us consider the restrictions on reduction mentioned in Appendix B.1. The unified β -redex takes the form $b(Q_1).b\lambda z:Q_2$, while ζ -reduction is allowed on the items $+\delta z\leftarrow Q$ and not allowed on the items $-\delta z\leftarrow Q$.

C. A NOTE ON THE CURRENT STATE OF THE FORMAL SPECIFICATION

In this appendix we discuss the current state of the definitions that formally specify $\chi\lambda\delta$ in the Calculus of Inductive Constructions [Guidi 2007a] in terms of modifications with respect to their initial state [Guidi 2006].

Firstly we set up a mechanism to avoid the need of exchanging the environment binders in the proof of Theorem 6(8). In particular we defined an extension of the `lift` function and an extension of the `drop` function [Guidi 2006] that apply a finite number of relocations to a term. The “relocation parameters” (i.e. the arguments h and i of the `lift` function) are contained in a list of pairs (h, i) . Here \bar{s} will always denote a variable for such a list.

These definitions are given in Definition 31 and Definition 32 below.

DEFINITION 31 THE MULTIPLE RELOCATION FUNCTION.

$$\begin{cases} \uparrow_{\circ} T & \equiv T \\ \uparrow_{((h,i);\bar{s})} T & \equiv i \uparrow^h \uparrow_{\bar{s}} T \end{cases} \quad (19)$$

DEFINITION 32 AXIOMS FOR MULTIPLE DROPPING.

- (1) (*non recursive case*)
 $\downarrow_{\circ} C = C.$
- (2) (*recursive case*)
If $i \downarrow^h C_1 = C_2$ and $\downarrow_{\bar{s}} C_2 = C_3$ then $\downarrow_{((h,i);\bar{s})} C_1 = C_3.$

With these functions we were able to rephrase Definition 21 as follows:

DEFINITION 33 THE STRONG REDUCIBILITY PREDICATE.

$$\begin{cases} (C, T) \in [(h, k)]_g & \text{iff } C \vdash_g T \triangleright (h, k) \text{ and } \text{sn}(C, T) \\ (C, T) \in [L_1 \rightarrow L_2]_g & \text{iff } C \vdash_g T \triangleright L_1 \rightarrow L_2 \text{ and for each } D, W, \bar{s}, \\ & (D, W) \in [L_1]_g \text{ and } \downarrow_{\bar{s}} D = C \text{ imply } (D, (W) \cdot \uparrow_{\bar{s}} T) \in [L_2]_g \end{cases} \quad (20)$$

The other definitions not included in [Guidi 2006] were formalized substantially as they appear in the previous sections, and we omit them here.

Remarkably we made some corrections to the preorders on environments (Definition 19, Definition 20, Definition 22) in order to prove Theorem 10(3).

Notice that relocations (i.e. applications of the `lift` function) were added where necessary both in the definitions and the theorems because in [Guidi 2007a], variables are referenced by position and not by name as in the present paper.

Secondly we took a final decision about the notation of the cast item, for which we now use $\langle V \rangle$ instead of $\{V\}$ (see Definition 1, Definition 27 and Definition 29). We also changed the native type assignment rule Figure 7(cast) because the former version applies an τ -reduction at the level of types in contrast with the general policy stated in Subsection 1.2. Theorem 7(6) is changed accordingly.

Thirdly we took a final decision on the domain of the exclusion binder and we rearranged the overall architecture of the calculus, also inserting the block for declared constants (see Subsection 5.1 and Subsection 5.2).

At the same moment we took a final decision on the name of the extension of $\lambda\delta$ with the unconditioned exclusion binder, which is now $\chi\lambda\delta$ instead of $\lambda\delta\chi$.

Finally we used \rightarrow_{τ} here in place of \rightarrow_{ϵ} for the reduction step that removes explicit type casts to avoid a clash with other reduction steps named ϵ appearing in the literature (see for instance the calculus $\lambda\epsilon$ in [Sørensen and Urzyczyn 2006]).

Currently (May 2008), the *Basic* module of the certified specification [Guidi 2007a] consists of 525 kilobytes of COQ vernacular describing 85 definitions and 683 theorems. The *Ground* module, that extends the standard library of COQ, consists of 34 kilobytes of vernacular describing 28 definitions and 50 theorems. From the standard library of COQ we borrow 18 definitions and 69 theorems.

D. POINTERS TO THE CERTIFIED PROOFS

As we mentioned in Subsection 1.3 the certified proofs of all results stated in this paper are available as resources of the Hypertextual Electronic Library of Mathematics (HELM) and their representation in natural language can be obtained through the HELM rendering software. Each proof is identified by a path that we list below.

We provide two methods to obtain the representation of a proof:

- The **dynamic representation** is generated on the fly by the HELM rendering software, which is very slow when big proofs are rendered (such as Theorem 3(1)). Visit the HELM on-line library at <http://helm.cs.unibo.it/browse/>, follow the path `matita/lambda-delta/plain/Basic/` and then the path of the proof. You can not reach a proof by concatenating these paths in a single `http` address.
- The **static representation** has been already generated so it displays faster. Visit the $\lambda\delta$ web site at <http://helm.cs.unibo.it/lambda-delta/static/>, follow the path `matita/lambda-delta/plain/Basic/` and then the path of the proof, that in this case has `.html` appended at the end. You can also reach the proof by concatenating these three paths in a single `http` address.

The proofs are displayed correctly only selecting a font with UNICODE support.

The following paths are parts of Uniform Resource Identifiers (URI) [Network Working Group 1998] so we can not guarantee their persistence.

- (1) Path for Theorem 1(1): `arity/props/node_inh.con`
- (2) Path for Theorem 1(2): `arity/props/arity_mono.con`
- (3) Path for Theorem 1(3): `arity/subst0/arity_fsubst0.con`
- (4) Path for Theorem 1(4): `csuba/arity/csuba_arity.con`
- (5) Path for Theorem 2(1): `arity/pr3/arity_sred_wcpr0_pr0.con`
- (6) Path for Theorem 2(2): `arity/pr3/arity_sred_pr3.con`
- (7) Path for Theorem 3(1): `pr0/props/pr0_subst0.con`
- (8) Path for Theorem 3(2): `pr0/pr0/pr0_confluence.con`
- (9) Path for Theorem 3(3): `pr3/pr3/pr3_confluence.con`
- (10) Path for Theorem 3(4): `pc3/props/pc3_thin_dx.con`
- (11) Path for Theorem 3(5): `pc3/props/pc3_head_1.con`
- (12) Path for Theorem 3(6): `pc3/props/pc3_head_2.con`
- (13) Path for Theorem 3(7): `pc3/fwd/pc3_gen_abst.con`
- (14) Path for Theorem 3(8): `pc3/props/pc3_eta.con`
- (15) Path for Theorem 4: `nf2/arity/arity_nf2_inv_all.con`
- (16) Path for Theorem 5(1): `csubc/csuba/csubc_csuba.com`
- (17) Path for Theorem 5(2): `csubc/arity/csubc_arity_conf.com`

- (18) Path for Theorem 6(1): `sn3/nf2/sn3_nf2.con`
- (19) Path for Theorem 6(2): `sc3/props/sc3_cast.con`
- (20) Path for Theorem 6(3): `sc3/props/sc3_abbr.con`
- (21) Path for Theorem 6(4): `sc3/props/sc3_abst.con`
- (22) Path for Theorem 6(5): `sc3/props/sc3_sn3.con`
- (23) Path for Theorem 6(6): `sc3/props/sc3_bind.con`
- (24) Path for Theorem 6(7): `sc3/props/sc3_appl.con`
- (25) Path for Theorem 6(8): `sc3/arity/sc3_arity_csubc.con`
- (26) Path for Theorem 6(9): `sc3/arity/sc3_arity.con`
- (27) Path for Theorem 7(1): `ty3/fwd/ty3_gen_sort.con`
- (28) Path for Theorem 7(2): `ty3/fwd/ty3_gen_lref.con`
- (29) Path for Theorem 7(3): `ty3/fwd/ty3_gen_bind.con`
- (30) Path for Theorem 7(4): `ty3/fwd/ty3_gen_bind.con`
- (31) Path for Theorem 7(5): `ty3/fwd/ty3_gen_appl.con`
- (32) Path for Theorem 7(6): `ty3/fwd/ty3_gen_cast.con`
- (33) Path for Theorem 8(1): `ty3/props/ty3_lift.con`
- (34) Path for Theorem 8(2): `ty3/props/ty3_correct.con`
- (35) Path for Theorem 8(3): `ty3/props/ty3_unique.con`
- (36) Path for Theorem 8(4): `ty3/fsubst0/ty3_fsubst0.con`
- (37) Path for Theorem 8(5): `ty3/fsubst0/ty3_subst0.con`
- (38) Path for Theorem 8(6): `ty3/fsubst0/ty3_csubst0.con`
- (39) Path for Theorem 8(7): `csubt/ty3/csubt_ty3.con`
- (40) Path for Theorem 8(8): `ty3/props/ty3_typecheck.con`
- (41) Path for Theorem 9(1): `ty3/pr3/ty3_sred_wcpr0_pr0.con`
- (42) Path for Theorem 9(2): `ty3/pr3/ty3_sred_pr3.con`
- (43) Path for Theorem 9(3): `ty3/pr3_props/ty3_gen_lift.con`
- (44) Path for Theorem 9(4): `ty3/pr3_props/ty3_tred.con`
- (45) Path for Theorem 9(5): `ty3/pr3_props/ty3_sconv_pc3.con`
- (46) Path for Theorem 9(6): `ty3/pr3_props/ty3_sconv.con`
- (47) Path for Theorem 10(1): `ty3/arity/ty3_arity.con`
- (48) Path for Theorem 10(2): `ty3/arity_props/ty3_sn3.con`
- (49) Path for Theorem 10(3): `csubt/csuba/csubt_csuba.com`
- (50) Path for Theorem 10(4): `ty3/arity_props/ty3_predicative.con`
- (51) Path for Theorem 10(5): `ty3/arity_props/ty3_repellent.con`
- (52) Path for Theorem 10(6): `ty3/arity_props/ty3_acyclic.con`
- (53) Path for Theorem 11(1): `pc3/dec/pc3_dec.con`
- (54) Path for Theorem 11(2): `ty3/dec/ty3_inference.con`
- (55) Path for Theorem 12(1): `ty3/sty0/ty3_sty0.con`
- (56) Path for Theorem 12(2): `sty1/env/sty1_env.con`
- (57) Path for Theorem 13(1): `ex0/props/leqz_leq.con`

- (58) Path for Theorem 13(2): `ex0/props/leq_leqz.con`
- (59) Path for Theorem 14(1): `ex1/props/ex1_arity.con`
- (60) Path for Theorem 14(2): `ex1/props/ex1_ty3.con`
- (61) Path for Theorem 15(1): `ex2/props/ex2_nf2.con`
- (62) Path for Theorem 15(2): `ex2/props/ex2_arity.con`
- (63) Path for Theorem 16(1): `pc3/props/pc3_head_2.con`
- (64) Path for Theorem 16(2): `sc3/props/sc3_bind.con`
- (65) Path for Theorem 16(3): `ty3/fwd/ty3_gen_bind.con`
- (66) Path for Theorem 17(1): `wf3/props/wf3_total.con`
- (67) Path for Theorem 17(2): `wf3/ty3/wf3_ty3_conf.con`
- (68) Path for Theorem 17(3): `wf3/props/wf3_ty3.con`

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