A. RELATED WORK

Below we review related work and compare it with our MWeb framework and semantics. A detailed comparison of the MWebAS and MWebWFS semantics of modular rule bases, as presented here, with an older version [Analyti et al. 2008] is provided at the end of this Section.

The concept of local CWAs was first introduced in [Etzioni et al. 1997]. In that work, the semantics of a local CWA about a sentence Φ appearing in an information source s is that: for all variable substitutions θ, if the ground sentence Φθ is true in the world then Φθ is entailed by s. Thus, any ground sentence Φθ that is not entailed by s is known to be false. Formally: 

$$\text{LCWA}(\Phi) \equiv (s \models \Phi\theta) \lor (s \models \neg \Phi\theta).$$

As seen in Proposition 7.13, this notion of local CWA also appears in our framework, in the case that a predicate p is declared freely closed in a rule base s and p is c-stratified in s w.r.t. a modular rule base S, where θ is a variable substitution from HU_S.

In [Heflin and Munoz-Avila 2002], local CWAs, with the semantics defined in [Etzioni et al. 1997], are applied for agent planning in the Semantic Web. In particular, [Heflin and Munoz-Avila 2002] considers independent information sources over the web, containing (i) knowledge, expressed in DAML+OIL [Horrocks et al. 2002] or the SHOE language [Heflin et al. 2003], and (ii) explicit local CWAs. If an agent needs information about a predicate p that is not contained in its knowledge base then p is passed to the Semantic Web Mediator of the agent. The Semantic Web Mediator queries relevant information sources about p and stops if: (i) an answer is found, or (ii) the local CWAs of a source s determine that s has complete...
knowledge about p. In that work, strong negation is not considered and modularity issues are rather trivial, as information sources do not interact.

A form of local CWA w.r.t. a context is proposed in [Cortés-Calabuig et al. 2005], where the local CWA is applied on the predicates of a single data source s, containing only positive facts. In that work, a context is a first-order formula over the predicates of s. The semantics of the proposed local CWA syntax is defined in first-order logic. Rules, strong negation, and modularity issues are not considered, in that work.

An alternative proposal for local CWAs is present in the dlvhex system [Eiter et al. 2005]. This answer-set programming system has features, like high-order atoms and external atoms, which are very flexible. For instance, assuming that a generic external atom KB[C](X) is available for querying a concept C in a knowledge base KB then a CWA can be stated as follows: C′(X) ← concept(C), concept(C′), cwa(C, C′), o(X), ¬KB[C](X), where concept(C) is a predicate which holds for all concepts C, the predicate cwa(C, C′) states that C′ is the complement of C under the closed world assumption, and o(X) is a predicate that holds for all individuals occurring in KB. Strong negation and modularity issues are not considered, in that work.

The combination of open-world and closed-world reasoning, in the same framework, is also proposed in [Analyti et al. 2008], where the stable model semantics of Extended RDF (ERDF) ontologies is developed. Intuitively, an ERDF ontology is the combination of (i) an ERDF graph G containing (implicitly existentially quantified) positive and negative information, and (ii) an ERDF program P containing derivation rules, with possibly all connectives ¬, ⊨, ⊃, ∧, ∨, ∀, ∃ in the body of a rule, and strong negation ¬ in the head of a rule. However, modularity issues are not considered there, and local CWAs and OWAs are not declared w.r.t. a context.

Flora-2 [Yang et al. 2003] is a rule-based object-oriented knowledge base system for reasoning with semantic information on the Web. It is based on F-logic [Kifer et al. 1995] and supports metaprogramming, non-monotonic multiple inheritance, logical database updates, encapsulation, modules with dynamically assigned content, and qualified literals. Module indicators in qualified literals can be module names or variables that get bound to a module name at run time. In Flora-2, reasoning mode and predicate scope issues are ignored. Additionally, strong negation is not supported. Simple literals appearing in a file, that is loaded to a module, are assumed to be qualified by the module name. The semantics of a modular rule base S is defined, based on the F-logic semantics [Kifer et al. 1995] of an equivalent rule base with no modules. In particular, each qualified atom subject[predicate ← object]@Nam_s (where Nam_s is a module name) is translated to subject[predicate#Nam_s ← object], where predicate#Nam_s is a new predicate name.

TRIPLE [Sintek and Decker 2002] is a rule language for the Semantic Web that supports modules (called, models there), qualified literals, and dynamic module transformation. Its syntax is based on F-Logic [Kifer et al. 1995] and supports a fragment of RDFS and first-order logic. All variables must be explicitly quantified, either existentially or universally. Arbitrary formulas can be used in the body, while the head of the rules is restricted to atoms or conjunctions of molecules. Module
indicators in qualified literals can be module names, variables, or skolem functions, as well as conjunction and difference of module indicators. However, the latter two cases do not add expressive power, as they can be equivalently defined through qualified literal conjunctions and the use of weak negation. The semantics of a modular rule base is defined based on the well-founded semantics (WFS) [Gelder et al. 1991] of an equivalent logic program. In that work, reasoning mode, predicate scope, and visibility issues are ignored. Additionally, strong negation is not supported.

In [Pontelli et al. 2006], a modularity framework for rule bases is proposed and its AS semantics is defined. However, in that work, the dependency graph $G$ between the rule bases of a modular rule base $S$ (formed based on the rule bases’ import statements) should be acyclic, facilitating distributed evaluation. The answer sets of a module $s \in S$ w.r.t. $S$ are defined based on the answer sets of the modules that are lower than $s$ in the dependency graph $G$. In that work, reasoning mode and predicate scope issues are ignored. Additionally, strong negation is not supported.

Another modularity framework for rule bases is proposed in [Polleres et al. 2006], where weakly negated rule literals should be qualified and depend (directly or indirectly) on qualified literals, only. In that work, reasoning mode, predicate scope, and visibility issues are ignored. Additionally, strong negation is not supported. The contextually bounded AS and contextually bounded WFS semantics of a modular rule base $S$ are defined, through the AS and WFS semantics of an equivalent logic program $S_{CB}$. $S_{CB}$ consists of the rules of each rule base $s \in S$ (called contexts, there) union the rules $p_{\text{Nam}_s}(t_1, \ldots, t_n) \leftarrow \text{Body}$, where $p(t_1, \ldots, t_n) \leftarrow \text{Body}$ is a rule defined in a rule base $s \in S$. Another proposal made in [Polleres et al. 2006] is to qualify all simple atoms appearing in a rule base $s$ by the name of $s$. The resulting rules union the original rules of each rule base $s \in S$ form a normal logic program $S_{CC}$. Then, the contextually closed AS and contextually closed WFS semantics of a modular rule base $S$ are defined through the AS and WFS semantics of $S_{CC}$.

Modularity for rule bases is also considered in [Brewka et al. 2007], where the contextual AS and the contextual WFS semantics of a modular rule base $S$ are defined model-theoretically. However, in that work, reasoning mode, predicate scope, and visibility issues are ignored. Simple literals appearing in a rule base $s$ (called context, there) are assumed to be qualified by the name of $s$. Intuitively, we can say that, if (i) all predicates of the rule bases in $S$ are defined in normal reasoning mode, (ii) all literals appearing in the body of the rules of the rule bases in $S$ are qualified, (iii) predicate scope and visibility issues are ignored, and (iv) $D_{S, m}^S = S$ then the MWebAS and contextual AS semantics of a rule base $s \in S$ coincide. However, this property is not true for MWebWFS and contextual WFS. Indeed, in contrast to MWebWFS, contextual WFS is not coherent. Further, we would like to note that since in our theory, in general $D_{S, m}^S \subseteq S$, for an $s \in S$ and $m \in \{d, o, c, n\}$, it is possible that $s$ has a normal answer set in reasoning mode $m$ w.r.t. $S$, even though $S$ does not have a stable contextual model, according to contextual AS semantics.

We want to note that all modularity frameworks in [Pontelli et al. 2006; Polleres App–3]
et al. 2005; Brewka et al. 2007] achieve monotonicity of reasoning in the case that a modular rule base $S$ is expanded with additional rule bases (while original rule bases in $S$ remain the same). Our framework also achieves this kind of monotonicity, as expressed in Proposition 7.5. However, our framework achieves also a more general kind of monotonicity for global predicates that is described in Theorem 7.9.

In [Aßmann et al. 2007], a general framework for modules in rule-based languages is proposed, which does not adhere to a particular rule language and semantics. Each module consists of a set of private rules, a set of public rules, and a set of dependency relationships between a body part of a rule and a head part of another. The authors propose a single algebraic operation, applied to considered modules, until a single module is derived. This is the scoped import operation $s \times_S s'$, where $S$ is a set of pointers to body parts of rules in $s'$. In particular, $s \times_S s'$ makes a copy $s''$ of $s$ and transfers all the rules of $s'$ to the private rules of $s''$, while dependency relationships are readjusted. For our case, where deductive rules are considered, the dependency relationships between the rule parts of a module are derived dynamically, based on predicate names. Note that, as scoped import transfers all rules from the operand modules to the resulting module, the existence of a global model is a requirement. Further, in that work, rules (and not defined predicates) are declared as public or private.

A framework for modular logic programs is also considered in [Eiter et al. 1997], where each module is seen as a generalized quantifier and is associated with a logic program, a set of input predicates, and a single output predicate. The output predicate of a module $s$ actually corresponds to a local predicate in our framework, while other predicates defined in $s$ correspond to internal predicates. Similarly, the input predicates of a module $s$ correspond to used predicates in our framework. A module can be called from a main logic program or from other modules. However, the respective call graph between modules should be acyclic. Weak negation is considered and defined semantics extends the stable model semantics [Gelfond and Lifschitz 1988]. However, strong negation is not considered, reasoning mode issues are ignored, while the visibility mechanism is simple.

A related direction of research is on modularity frameworks for Description Logic (DL) ontologies [Baader et al. 2003] (for an overview and qualitative comparison of these frameworks, see [Grau and Kutz 2007; Bao et al. 2006]). In E-connections for DLs [Kutz et al. 2004], a set of DL ontologies $s_1, \ldots, s_n$ with disjoint vocabularies is connected via a set of link relations $\mathcal{E} = \bigcup_{1 \leq j \leq n} E_{i,j}$. Link relations in $E_{i,j}$ are used in $s_i$ for forming the concepts: $\exists E_{i,j} C_j$ and $\forall E_{i,j} C_j$, where $C_j$ is a concept of $s_j$. In Distributed Description Logics (DDLs) [Borgida and Serafini 2003; Serafini et al. 2005], a set of DL ontologies $s_1, \ldots, s_n$ with disjoint vocabularies is connected via (i) a set of bridge rules of the form $C_i \sqsubseteq C_j$ or $C_i \sqsupseteq C_j$, where $C_i$ and $C_j$ are concepts of $s_i$ and $s_j$, respectively, (ii) a set of partial individual correspondences $a_i \mapsto b_j$, where $a_i$ and $b_j$ are objects of $s_i$ and $s_j$, respectively, and (iii) a set of complete individual correspondences $a_i \mapsto \{ b_1^j, \ldots, b_n^j \}$, where $a_i$ is an object of $s_i$ and $b_1^j, \ldots, b_n^j$ are objects of $s_j$. The problem of local inconsistency polluting the inferences of all modules in a modular representation is handled in the version of DDL [Serafini et al. 2005] through a special interpretation, called hole, whose role is to interpret even inconsistent local T-boxes. In our work, we achieve the same
result since the semantics of a rule base \( s \) in reasoning mode \( m \) w.r.t. \( S \) depends only on the pairs of rules bases \( s' \) and reasoning modes \( x, (s', x) \in D_{\text{BT}}^S \) (and not on all pairs in \( S \times \{d, o, c, n\} \)). In Package-based Description Logics (P-DL) [Bao et al. 2006], a set of DL ontologies \( s_1, ..., s_n \) (called packages) is connected through the use of common terms. Each term and axiom is associated with a single home package. A package \( s_i \) may use foreign terms, that is terms whose home is another package \( s_j \).

Despite their differences, modular DL ontologies and \( \mathbb{M} \text{Web} \) modular rule bases share some common goals, such as encapsulation, localized semantics, partial knowledge reuse, and directed semantic relations. Yet, there exist some major differences between the two approaches. Modular rule bases share a common Herbrand Universe, whereas each DL module has its own local domain of interpretation. Additionally, although DL ontologies are more expressive than rules in certain aspects [Grosof et al. 2003], rules provide for more expressive forms of module interconnection. Moreover, \( \mathbb{M} \text{Web} \) modular rule bases provide full support for negation (weak and strong).

Finally, we would like to mention a general framework for multi-context reasoning, proposed in [Brewka and Eiter 2007], that allows to combine arbitrary monotonic and non-monotonic logics. Information flow between the different contexts is achieved through a set of non-monotonic bridge rules. In that work, several notions for acceptable belief states for the multicontext system are investigated.

As a fair account, we should say that though many of the previously described works do not support strong negation, this can be added as an easy extension of their theories, for Answer Set based semantics. The treatment of strong negation in well-founded based semantics is not so easy and immediate.

A.1 Comparison with older version of the \( \mathbb{M} \text{WebAS} \) and \( \mathbb{M} \text{WebWFS} \) semantics of modular rule bases

Below, we compare the \( \mathbb{M} \text{WebAS} \) and \( \mathbb{M} \text{WebWFS} \) semantics of modular rule bases, as presented here, with an older version [Analyti et al. 2008]. In [Analyti et al. 2008], a normal (resp. extended) interpretation of a modular rule base \( S \) is defined as a set \( M = \{ M_x^s | s' \in S, x \in \{d, o, c, n\} \} \), where \( M_x^s \) is a normal (resp. extended) interpretation of rule base \( s' \) w.r.t. \( S \). The minimal model of \( S \) w.r.t. a normal (resp. extended) interpretation of \( S \) is the least (w.r.t. \( \leq_1 \)) normal (resp. extended) interpretation of \( S \), \( M = \{ M_x^s | s' \in S \) and \( x \in \{d, o, c, n\} \} \), that satisfies the conditions of Definition 5.11, with the difference that: (i) in 5.11.1 and 5.11.2(b), symbol \( = \) is replaced by \( \geq \), and (ii) in 5.11.2(a) and 5.11.2(b), expression \( y = \text{least}(x, \text{mode}^y_x(p), |\text{mode}^y_x(p)|) \) is replaced by \( y = \text{least}(x, \text{mode}^y_x(p)) \). Then, a normal (resp. extended) answer set of \( S \) is defined as a normal (resp. extended) interpretation of \( S \) satisfying the condition that \( M = \text{least}(S, M) \) (resp. \( M = \text{Coh}(\text{least}(S, M)) \)).

As the following two examples demonstrate, it is possible that a rule base \( s \in S \) has a consistent normal (resp. extended) answer set at a reasoning mode \( x \in \{d, o, c, n\} \) w.r.t. \( S \), even though (in the old version), (i) in every normal (resp. extended) interpretation of the whole modular rule base, and not for a single rule base.

\(^{17}\)Note that in [Analyti et al. 2008], a normal (resp. extended) interpretation is defined for the whole modular rule base, and not for a single rule base.
answer set \( M \) of \( S \), \( M^x_\alpha \) is inconsistent, or \( S \) has no normal (resp. extended) answer set.

**Example 25.** Consider the modular rule base \( S = \{ s_1, s_2 \} \) of Figure 4. Note that, there is a single (consistent) normal (resp. extended) answer set of \( s_1 \) in reasoning mode \( n \) w.r.t. \( S \). Thus, \( s_1 \models_{S}^{\text{AS}} p(c) \) and \( s_1 \not\models_{S}^{\text{AS}} \neg p(c) \). However, according to the old version of \( \text{MWeb} \) semantics, in the single normal (resp. extended) answer set of \( S \), \( M \), it holds that \( M^x_\alpha \) is inconsistent. Thus, in the old version, it holds that \( s_1 \models_{\text{SEM}}^{\text{AS}} p(c) \) and \( s_1 \not\models_{\text{SEM}}^{\text{AS}} \neg p(c) \), for \( \text{SEM} \in \{ \text{mAS, mWFS} \} \). \( \square \)

<table>
<thead>
<tr>
<th>Rule base ( s_1 )</th>
<th>Rule base ( s_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(<a href="http://example1.org">http://example1.org</a>)</td>
<td>(<a href="http://example2.org">http://example2.org</a>)</td>
</tr>
<tr>
<td>defines local normal ex:p.</td>
<td>defines global definite ex:q</td>
</tr>
<tr>
<td>uses normal ex:q.</td>
<td>defines local normal ex:r</td>
</tr>
<tr>
<td>( p(c) \leftarrow q(c) ).</td>
<td>( q(c) ).</td>
</tr>
<tr>
<td></td>
<td>( \neg r(c) ).</td>
</tr>
<tr>
<td></td>
<td>( r(c) ).</td>
</tr>
</tbody>
</table>

Fig. 4. A modular rule base \( S \)

**Example 26.** Consider again the modular rule base \( S = \{ s_1, s_2 \} \) of Figure 4, with the difference that the fact “\( r(c) \leftarrow \neg q(d) \).” in rule base \( s_2 \) is replaced by the rule “\( r(c) \leftarrow \neg r(c) \).” Note that, there is a single (consistent) normal (resp. extended) answer set of \( s_1 \) in reasoning mode \( n \) w.r.t. \( S \). Thus, \( s_1 \models_{S}^{\text{AS}} p(c) \) and \( s_1 \not\models_{S}^{\text{AS}} \neg p(c) \). However, according to the old version of \( \text{MWeb} \) semantics, there is no normal (resp. extended) answer set of \( S \). Thus, in the old version, it holds that \( s_1 \models_{\text{SEM}}^{\text{AS}} p(c) \) and \( s_1 \not\models_{\text{SEM}}^{\text{AS}} \neg p(c) \), for \( \text{SEM} \in \{ \text{mAS, mWFS} \} \). Similar is the case if the fact “\( r(c) \).” in rule base \( s_2 \) is replaced by the rule “\( r(c) \leftarrow \neg r(c) \).” \( \square \)

The old version of \( \text{MWeb} \) semantics achieves monotonicity of reasoning in the case that a modular rule base \( S \) is expanded, if the set of rule bases from which knowledge about a predicate is imported in any rule base \( s \in S \) stays the same, after the expansion of \( S \). However, it does not achieve monotonicity of reasoning for global predicates in the more general case of modular rule base extension, as expressed in Theorem 7.9. This is demonstrated in the following example.

**Example 27.** Consider the modular rule base \( S = \{ s \} \) of Figure 5 at times \( t \) and \( t + 1^{18} \). Note that, at time \( t \), modular rule base \( S \) has a single (consistent) normal answer set (according to the old version of \( \text{MWebAS} \) semantics), \( M \), with \( M^x_\alpha(p(c)) = 1 \). Thus, at time \( t \), it holds \( s \models_{S}^{\text{AS}} p(c) \). However, at time \( t + 1 \), modular rule base \( S \) has two (consistent) normal answer sets (according to the old version of \( \text{MWebAS} \) semantics), \( M \) and \( N \), with \( M^x_\alpha(p(c)) = 1 \) and \( N^x_\alpha(\neg p(c)) = 1 \). Thus, at time \( t \), it holds \( s \not\models_{S}^{\text{AS}} p(c) \). \( \square \)

\(^{18}\)Note that at time \( t + 1 \), an additional rule is added to rule base \( s \).
Specifically, the logic program $P/I$ is obtained from $[P]$ as follows: (i) we remove from $[P]$, all rules that contain in their body a default literal $\neg L$, s.t. $I(L) = 1$, and (ii) we remove from the body of the remaining rules, any default literal $\neg L$, s.t. $I(L) = 0$. We say that $I$ is an answer set of $P$ if $I = \text{least}_3^T(P/I)$ [Gelfond and Lifschitz 1990; 1991].

Let $P$ be an ELP and let $I$ be a 3-valued interpretation of $P$. We define $\text{Co}h(I) = I \cup \{\neg L \mid L \in \mathbb{B}_P \text{ and } \neg L \in I\}$\textsuperscript{20}. Additionally, we denote by $P/\mathbb{WFSX}I$ the $P/I$ modulo operator of the WFSX semantics [Pereira and Alferes 1992; Alferes and Pereira 1996]. Specifically, the logic program $P/\mathbb{WFSX}I$ is obtained from $[P]$ as follows:

\begin{itemize}
  \item\textup{B}. PROOFS
\end{itemize}

Here, we prove the Propositions, Theorems, and Corollaries presented in the main paper. First, we provide a few definitions from the AS semantics [Gelfond and Lifschitz 1990; 1991] and the WFSX semantics [Pereira and Alferes 1992; Alferes and Pereira 1996], adjusted properly to fit our MWeb framework definitions.

Let $P$ be an ELP. A 2-valued interpretation of $P$ is a set $I$, where $I \subseteq \mathbb{B}_P$ s.t. either: $I \cap \neg I = \emptyset$ (consistency), or $I = \mathbb{B}_P$. A 3-valued interpretation of $P$ is a set $I = T \cup \neg F$, where $T, F \subseteq \mathbb{B}_P$ s.t. either: (i) $T \cap \neg T = \emptyset$ and $T \cap F = \emptyset$ (consistency), or (ii) $T = F = \mathbb{B}_P$.\textsuperscript{19}

Let $P$ be an ELP. We denote by $[P]$ the ground version of $P$, that is $P$ instantiated over all constants appearing in $P$. Let $I$ be a 2-valued (resp. 3-valued) interpretation of $P$. We define: $I \models P$ iff for all $r \in [P]$, it holds that $I(\text{Head}_r) \geq I(\text{Body}_r)$. Let $I, J$ be two 2-valued (resp. 3-valued) interpretations of $P$. We define: $I \preceq J$ iff for each $L \in \mathbb{B}_P$, it holds that $I(L) \leq J(L)$. Additionally, we define $I \preceq_t J$ iff $I \subseteq J$. A definite rule is an ELP rule without weak negation. Additionally, a definite logic program is an ELP with definite rules.

Let $P$ be a definite logic program. We define $\text{least}_3^T(P)$ to be the minimal (w.r.t. $\leq_2$) 2-valued interpretation $I$ s.t. $I \models [P]$. Similarly, we define $\text{least}_3^T(P)$ to be the minimal (w.r.t. $\leq_3$) 3-valued interpretation $I$ s.t. $I \models [P]$.

Let $P$ be an ELP and let $I$ be a 2-valued interpretation of $P$. We denote by $P/\mathbb{AS}I$ the Gelfond-Lifschitz $P/I$ modulo operator [Gelfond and Lifschitz 1990]. Specifically, the logic program $P/\mathbb{AS}I$ is obtained from $[P]$ as follows: (i) we remove from $[P]$, all rules that contain in their body a default literal $\sim L$, s.t. $I(L) = 1$, and (ii) we remove from the body of the remaining rules, any default literal $\sim L$, s.t. $I(L) = 0$. We say that $I$ is an answer set of $P$ if $I = \text{least}_3^T(P/\mathbb{AS}I)$ [Gelfond and Lifschitz 1990; 1991].

Let $P$ be an ELP and let $I$ be a 3-valued interpretation of $P$. We define $\text{Co}h(I) = I \cup \{\sim L \mid L \in \mathbb{B}_P \text{ and } \sim L \in I\}$\textsuperscript{20}. Additionally, we denote by $P/\mathbb{WFSX}I$ the $P/I$ modulo operator of the WFSX semantics [Pereira and Alferes 1992; Alferes and Pereira 1996]. Specifically, the logic program $P/\mathbb{WFSX}I$ is obtained from $[P]$ as follows:

\begin{itemize}
  \item\textup{Note that in contrast to [Pereira and Alferes 1992; Alferes and Pereira 1996], our definition of a 3-valued interpretation of $P$ includes also inconsistent and incoherent interpretations. We prefer this definition because it is more suitable for our proofs.}
  \item\textup{Note that if $L \in \mathbb{B}_P$ then $\neg(\neg L) = L$.}
\end{itemize}
(i) we remove from $[P]$, all rules that contain in their body an objective literal $L$ s.t. $I(¬L) = 1$ or a default literal $¬L$ s.t. $I(L) = 1$, (ii) we remove from the body of the remaining rules, any default literal $¬L$ s.t. $I(L) = 0$, and (iii) we replace all remaining default literals $¬L$ with $u$. We say that $I$ is a partial stable model$^{21}$ of $P$ iff $I = \text{Coh}(\text{least}_{I}^{2}(P/\text{ASP}))$ [Pereira and Alferes 1992; Alferes and Pereira 1996].

Finally, we note that many of the proofs use Theorem 6.1 and Corollary 6.3, provided in Section 6. Additionally, they use the notation $N_{M}$, defined above Theorem 6.1.

**Proposition 5.12** Let $N$ be a normal (resp. extended) interpretation of $s$ in reasoning mode $m$ w.r.t. $S$. It always exists the minimal model of $s$ in reasoning mode $m$ w.r.t. $S$ and $N$.

**Proof:** Let $N$ be a normal interpretation of $s$ in reasoning mode $m$ w.r.t. $S$. First, we will show that there exists a normal interpretation of $s$ in reasoning mode $m$ w.r.t. $S_{M}$, $M_{1}$ that satisfies the conditions (1-3) of Definition 5.11. Let $N = \text{least}_{I}^{2}(M_{S}^{m}/\text{ASP}N_{M})$. Additionally, let $M_{N}$ be the normal interpretation of $s$ in reasoning mode $m$ w.r.t. $S$ s.t. $N_{M} = N$. Then, $M$ satisfies the conditions (1-3) of Definition 5.11. In particular, $M$ satisfies conditions (1,2) of Definition 5.11 due to the rules generated (in Section 6) from the translation of the defines declarations of the rule bases $s'$, where $s' \in S$. Additionally, let $x \in \{d, o, c, n\}$ s.t. $N_{M} = N_{M}$. Then, $N_{M} >_{t} N_{M}$. However, this is impossible. Thus, $N_{M}$ is the minimal model of $s$ in reasoning mode $m$ w.r.t. $S$ and $N$.

The case that $N$ is an extended interpretation of $s$ in reasoning mode $m$ w.r.t. $S$ is proved similarly to the previous case. □

**Proposition 5.15** Let $M$ be a consistent normal or extended interpretation of rule base $s$ in reasoning mode $m$ w.r.t. $S$. Let $s' \in S$ and let $x \in \{d, o, c, n\}$ s.t. $\langle s', x \rangle \in D_{s,a}^{S}$. Additionally, let $M' = \{M_{s}', x \in M | \langle s'', y \rangle \in D_{s'', x}^{S}\}$. It holds that:

1. If $M \in \mathcal{M}_{n,S}^{m}(s)$ then $M' \in \mathcal{M}_{n,S}^{m}(s')$.
2. If $M \in \mathcal{M}_{n,S}^{m}(s)$ then $M' \in \mathcal{M}_{n,S}^{m}(s')$.

**Proof:**

1. We will denote $\Pi_{s}^{m}$ by $\Pi$ and $\Pi_{x}^{m}$ by $\Pi'$. Since $\Pi_{s}^{m} = \{\text{Head}_{s}^{m} \cup \text{Body}_{s}^{m} \subseteq D\}$ and $D_{s,a}^{\Pi} \subseteq D_{s,a}^{\Pi'},$ it follows that $\Pi' \subseteq \Pi$. We define $D = \Pi_{\Pi'}$.

Assume that $M \in \mathcal{M}_{n,S}^{m}(s)$. Then, based on Theorem 6.1, it holds that $N_{M} = \text{least}_{M}^{2}(\Pi/\text{ASP}N_{M})$. Further, it holds that if there exists $r \in [\Pi]$ with $\text{Head}_{r} \subseteq D$ then $\text{Body}_{r} \subseteq D$. It follows, from these facts, that $N_{M} \cap D = \text{least}_{M}^{2}(\Pi/\text{ASP}(N_{M} \cap D))$. Therefore, $N_{M} \cap D$ is an answer set of $\Pi'$. Since $N_{M} \subseteq N_{M} \cap D$, it follows that $N_{M}$ is an answer set of $\Pi'$. It follows now from this and Theorem 6.1 that $M' \in \mathcal{M}_{n,S}^{m}(s')$.

$^{21}$Note that we extend the definition of a partial stable model, that is provided in [Pereira and Alferes 1992; Alferes and Pereira 1996], to also include inconsistent partial stable models.
2) Let Π, Π′, and D be defined as in 1). Additionally, let \( D′ = D \cup \sim D \). Note that \( \Pi′ \subseteq \Pi \). Assume that \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \). Then, based on Theorem 6.1, it holds that \( N_M = \text{Coh}(\text{least}_n^\lambda((\Pi/\text{AF}) N_M)) \). Further, it holds that (i) if there exists \( r \in \Pi \) with \( \text{Head}_r \subseteq D \) then \( \text{Body}_r^+ \cup \text{Body}_r^- \subseteq D \), and (ii) if \( L \in D \) then any rule that defines \( L \) and \( \sim L \) in \( \Pi \) is found in \( \Pi′ \). It follows, from these facts, that \( N_M \cap D′ = \text{Coh}(\text{least}_n^\lambda((\Pi′/\text{AF}) (N_M \cap D′))) \). Since \( N_M′ = N_M \cap D′ \), it follows that \( N_M′ \) is a partial stable model of \( \Pi′ \). It follows now from this and Theorem 6.1 that \( M′ \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s′) \). □

**Proposition 5.16** If there exists \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) s.t. \( M \) is inconsistent then:

1. \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ M \} \), and
2. \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ M′ \} \), where \( M′ \) is inconsistent, or \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ \} \).

**Proof:**

1) We will denote \( \Pi_{s,S}^{\text{AS}} \) by \( \Pi \). Let \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) s.t. \( M \) is inconsistent. It follows from Theorem 6.1 that \( N_M \) is an inconsistent answer set of \( \Pi \). In [Gelfond and Lifschitz 1991], it is shown that if an extended logic program \( P \) has an inconsistent answer set then this is the only answer set of \( P \). Thus, \( N_M \) is the only answer set of \( \Pi \). Assume now that there exists \( M′ \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) such that \( M′ \) is consistent. Then, it follows from Theorem 6.1 that \( N_M′ \) is a consistent answer set of \( \Pi \), which is impossible. Thus, \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ M \} \).

2) Let \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) s.t. \( M \) is inconsistent. Assume that \( N \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) such that \( N \) is consistent. It follows from Theorem 6.1 that \( N_M = \text{Coh}(\text{least}_n^\lambda((\Pi/\text{AF}) N_M)) \). Note that \( \Pi/\text{AS} N_M \) contains all the definite rules of \( \Pi \). We will denote \( \Pi/\text{AS} N_M \) by \( P \) and the definite rules of \( \Pi \) that appear in \( \Pi/\text{AF} N_M \) by \( P′ \). Let \( \lambda \) be the least integer such that for any \( L \in \text{HB}_P \), it holds that \( L \in T_{P′}^\lambda(\emptyset) \) and \( \sim L \in N_M \). Assume that there exists such \( \lambda \). Then, all the rules in \( P \), applied in the derivation of \( L \), while computing \( T_{P′}^\lambda(\emptyset) \), appear in \( P′ \). Thus, \( L \in \Psi_{P′}^ω(\sim \text{HB}_P)^{\text{22}} \). Therefore, \( L \in \text{least}_n^\lambda((\Pi/\text{AF}) N_M) \). Since \( \text{least}_n^\lambda((\Pi/\text{AF}) N_M) \subseteq N_M \), it follows that \( L \in N_M \). Therefore, \( N_M \) is inconsistent. Thus, \( N \) is inconsistent, which is impossible. Assume now that there is no \( L \in \text{HB}_P \) s.t. \( L \in T_{P′}^\lambda(\emptyset) \) and \( \sim L \in N_M \). Since \( T_{P′}^\lambda(\emptyset) \) contains a pair of complementary literals, it follows that \( \Psi_{P′}^ω(\sim \text{HB}_P) \) contains also this pair of complementary literals. Thus, \( \text{least}_n^\lambda((\Pi/\text{AF}) N_M) \) is inconsistent. Therefore, \( N_M \) is inconsistent. Thus, \( N \) is inconsistent, which is impossible. Therefore, it holds that: (i) \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ M′ \} \), where \( M′ \) is inconsistent, or (ii) \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ \} \). □

**Proposition 5.18** If there exists \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) s.t. \( M \) is inconsistent then (i) rule base \( s \) is contradictory in reasoning mode \( m \) w.r.t. \( S \), and (ii) \( \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) = \{ M \} \).

**Proof:**

i) Let \( M \in \mathcal{M}_{\text{AS}_n,S}^{\text{EAS}}(s) \) s.t. \( M \) is inconsistent. Then, it follows from Theorem 6.1 that \( N_M \) is an inconsistent partial stable model of \( M \). We will denote \( \Pi_{s,S}^{\text{AS}} \) by \( P \) and \( P/\text{AF} N_M \) by \( P′ \). It holds that \( \Psi_{P′}^ω(\sim \text{HB}_P) \) contains a pair of complementary literals \( L \) and \( \sim L \). Let \( N = \text{HB}_P \) and \( P'' = P/\text{AS} \). Then, \( T_{P''}^ω(\emptyset) \) contains also

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22The \( \Psi_P \) operator is a generalization of the Van Emden-Kowalski least model operator \( T_P \) for definite logic programs \( P \), and is defined in [Aleres and Pereira 1996].
Let $M \in \mathcal{M}^\text{EAS}_s(s)$ s.t. $M$ is inconsistent. It follows from i) that $s$ is contradictory in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$. Therefore, it follows from Proposition 5.16 that $\mathcal{M}^\text{EAS}_s(s) = \{M\}$. □

**Proposition 5.19** Assume that rule base $s$ in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$ is contradictory. It holds that:

1) If $m \in \{d, o, c\}$ then rule base $s$ in reasoning mode $x \in \{d, o, c, n\}$ w.r.t. $\mathcal{S}$ is also contradictory.

2) If $s' \in \mathcal{S}$ and $x \in \{d, o, c, n\}$ s.t. $(s, m) \in D^S_{x, s}$ then rule base $s'$ in reasoning mode $x$ w.r.t. $\mathcal{S}$ is contradictory.

**Proof:**

1) Let $m \in \{o, c\}$ and assume that rule base $s$ in reasoning mode $\text{d}$ w.r.t. $\mathcal{S}$ is contradictory. We will show that rule base $s$ in reasoning mode $\text{d}$ w.r.t. $\mathcal{S}$ is also contradictory. Let $M$ be the inconsistent normal answer set of $s$ in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$. Then, it follows from Theorem 6.1 that $N_M$ is the inconsistent answer set of $\Pi^S_s$. Thus, $\text{least}_2^\text{e}(\Pi^S_s/\text{AS} N_M)$ is inconsistent. Note that $\Pi^S_s/\text{AS} N_M$ contains exactly all the definite rules in $\Pi^S_s$. It follows from the way $\Pi^S_s$ and $\Pi^S_s$ are defined that $\text{least}_2^\text{e}(\Pi^S_s)$ is also inconsistent. Therefore, $\Pi^S_s$ has an inconsistent answer set. It now follows from Theorem 6.1 that rule base $s$ has an inconsistent answer set in reasoning mode $\text{d}$ w.r.t. $\mathcal{S}$. Thus, rule base $s$ in reasoning mode $\text{d}$ w.r.t. $\mathcal{S}$ is contradictory.

Let $x \in \{o, c, n\}$ and assume that rule base $s$ in reasoning mode $\text{d}$ w.r.t. $\mathcal{S}$ is contradictory. It follows from Theorem 6.1 that $\text{least}_2^\text{e}(\Pi^S_s)$ is inconsistent. Let $N$ be the inconsistent 2-valued interpretation of $\Pi^S_s$ (i.e. $N = \mathbb{H}_P$, where $P = \Pi^S_s$). As $\Pi^S_s/\text{AS} N$ contains all the definite rules in $\Pi^S_s$ and from the way $\Pi^S_s$ and $\Pi^S_s$ are defined, it follows that $\text{least}_2^\text{e}(\Pi^S_s)$ is also inconsistent. Thus, $N$ is an answer set of $\Pi^S_s$. It now follows from Theorem 6.1 that rule base $s$ has an inconsistent answer set in reasoning mode $x$ w.r.t. $\mathcal{S}$. Thus, rule base $s$ in reasoning mode $x$ w.r.t. $\mathcal{S}$ is contradictory. Statement 1) now follows.

2) Assume that rule base $s'$ in reasoning mode $x$ w.r.t. $\mathcal{S}$ is not contradictory. Let $M$ be a consistent normal answer set of $s'$ in reasoning mode $x$ w.r.t. $\mathcal{S}$. Let $M' = \{M'^{s'}_n \in M | (s', y) \in D^S_{s, x}\}$. Obviously, $M'$ is consistent. It follows from Proposition 5.15 that $M'$ is an answer set of $s$ in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$. Since $M'$ is consistent and rule base $s$ in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$ is contradictory, this is impossible. □

**Proposition 5.20** It holds that: $|\text{minimal}_{\leq}(\mathcal{M}^\text{EAS}_s(s))| \leq 1$.

**Proof:** Assume that rule base $s$ has two different minimal (w.r.t. $\leq_k$) extended answer sets in reasoning mode $\text{m}$ w.r.t. $\mathcal{S}$, $M$ and $N$. It follows from Theorem 6.1 that $N_M$ and $N_M$ are two different partial stable models of $\Pi^S_s$. Assume that $N_M$ is not a minimal (w.r.t. $\leq_k$) partial stable model of $\Pi^S_s$. Thus, there exists a partial stable model of $\Pi^S_s$, $N_{M'}$, s.t. $N_{M'} \prec_k N_M$. But then, from Theorem 6.1, $M'$ is an
Let $S$ be a modular rule base, let $s \in S$, and let $m \in \{d, o, c, n\}$.

**Proposition 6.2** Let $S$ be a modular rule base, and let $m \in \{d, o, c, n\}$ s.t. $A_{m,S}^{\text{EAS}}(s) \neq \emptyset$. It holds that: $M = mW^S$ if $N_M$ is the well-founded model (according to WFSX semantics) of $\Pi^m_{s,S}$.

**Proof:** Since $A_{m,S}^{\text{EAS}}(s) \neq \emptyset$, it follows from Definition 5.21 that $M = \text{least}_m(A_{m,S}^{\text{EAS}}(s))$. It follows from Theorem 6.1 that $N_M$ is a partial stable model of $\Pi^m_{s,S}$. It holds that the well-founded model (according to WFSX semantics) of $\Pi^m_{s,S}$ is the least w.r.t. $\leq_k$.
partial stable model of $\Pi_{a,S}$ [Alferes and Pereira 1996]. Assume that $N_M$ is not the least w.r.t. $\leq_2$ partial stable model of $\Pi_{a,S}$ and that there exists a partial stable model of $\Pi_{a,S}$, $N_N$, s.t. $N_M \leq_2 N_N$. Then, it follows from Theorem 6.1 that $M \not\leq_2 N$ and $N \in \mathcal{M}_{\Pi_{a,S}}$($s$). However, this is impossible. Thus, $N_M$ is the well-founded model (according to $\text{WFSX}$ semantics) of $\Pi_{a,S}$.

$\Leftarrow$) Let $N_M$ be the well-founded model (according to $\text{WFSX}$ semantics) of $\Pi_{a,S}$. Then, $N_M$ is the least w.r.t. $\leq_2$ partial stable model of $\Pi_{a,S}$ [Alferes and Pereira 1996]. It follows from Theorem 6.1 that $M \in \mathcal{M}_{\Pi_{a,S}}$($s$). Assume that there exists $N \in \mathcal{M}_{\Pi_{a,S}}$($s$) s.t. $M \not\leq_2 N$. Then, it follows from Theorem 6.1 that $N_M$ is a partial stable model of $\Pi_{a,S}$ and $N_M \not\leq_2 N_N$, which is impossible. Thus, $M = \text{least}_{\leq_2}(\mathcal{M}_{\Pi_{a,S}}$($s$)). □

**Proposition 7.1** Let $\mathcal{S}$ be a modular rule base, let $s \in \mathcal{S}$, and let $L \in \text{HB}^S \cup \sim \text{HB}^S$. It holds that: if $s \models_{\text{HBS}} L$ then $s \models_{\text{HBS}} L$.

**Proof:** Let $\text{Pred}(L) = p$. If $p \in \text{Pred}^o_s$ and $\text{qual}(L)$ is undefined then let $m = |\text{mode}^o_s(p)|$. If $p \in \text{Pred}^D_s - \text{Pred}^o_s$ and $\text{qual}(L)$ is undefined then let $m = \text{mode}^D_s(p)$. If $p \in \text{Pred}^D_s$ and $\text{qual}(L)$ is defined then let $m = \text{mode}^D_s(p)$.

In the case that rule base $s$ is contradictory in reasoning mode $m$ w.r.t. $\mathcal{S}$ then it holds that $s \models_{\text{HBS}} L$. Assume that rule base $s$ is not contradictory in reasoning mode $m$ w.r.t. $\mathcal{S}$. Note that if $N$ is a consistent answer set of an ELP $P$ then $N \cup \sim (\text{HB}_P - N)$ is a partial stable model of $P$. Since rule base $s$ is not contradictory in reasoning mode $m$ w.r.t. $\mathcal{S}$, it follows from Theorem 6.1 that $\Pi_{a,S}$ has only consistent answer sets.

Assume now that $s \models_{\text{HBS}} L$. Then, it follows from Theorem 6.1 that for every partial stable model $M$ of $\Pi_{a,S}$, it holds that $M(\tau^*_s(L)) = 1$. Thus, it follows that for every answer set $M'$ of $\Pi_{a,S}$, it holds that $M'(\tau^*_s(L)) = 1$. Therefore, it follows from Theorem 6.1 that $s \models_{\text{HBS}} L$. □

**Proposition 7.2** Let $\mathcal{S}$ be a modular rule base, let $s \in \mathcal{S}$, and let $L \in \text{HB}^S \cup \sim \text{HB}^S$. It holds that: (i) the problem of establishing if $s \models_{\text{HBS}} L$ is data complete for co-NP and program complete for co-NEXPTIME and (ii) the problem of establishing if $s \models_{\text{HBS}} L$ is data complete for P and program complete for EXPTIME.

**Proof:** Let $\text{Pred}(L) = p$. If $p \in \text{Pred}^D_s$ and $\text{qual}(L)$ is undefined then let $m = |\text{mode}^D_s(p)|$. If $p \in \text{Pred}^D_s - \text{Pred}^o_s$ and $\text{qual}(L)$ is undefined then let $m = \text{mode}^D_s(p)$. If $p \in \text{Pred}^D_s$ and $\text{qual}(L)$ is defined then let $m = \text{mode}^D_s(p)$.

(i) It follows from Corollary 6.3 that $s \models_{\text{HBS}} L$ if $\tau^*_s(L) \in \mathcal{C}_{\text{AS}}(\Pi_{a,S})$. Note that there is a polynomial time transformation from $s, \mathcal{S}$ to $\Pi_{a,S}$. Let $P$ be an ELP and let $L' \in \text{HB}_P \cup \sim \text{HB}_P$. It is stated in [Dantsin et al. 2001] that the problem of establishing if $L' \in \mathcal{C}_{\text{AS}}(P)$ is data complete for co-NP and program complete for co-NEXPTIME. Thus, the problem of establishing if $s \models_{\text{HBS}} L$ has data complexity co-NP and program complexity co-NEXPTIME. Now due to Proposition 7.4, it follows that the problem of establishing if $s \models_{\text{HBS}} L$ has data complexity co-NP and program complete for co-NEXPTIME.

(ii) It follows from Corollary 6.3 that $s \models_{\text{HBS}} L$ if $\tau^*_s(L) \in \mathcal{C}_{\text{WFSX}}(\Pi_{a,S})$. Note that there is a polynomial time transformation from $s, \mathcal{S}$ to $\Pi_{a,S}$. Let $P$ be an ELP and let $L' \in \text{HB}_P \cup \sim \text{HB}_P$. It is shown in [Alferes and Pereira 1996] that the problem of establishing if $L' \in \mathcal{C}_{\text{WFSX}}(P)$ has data complexity P. Thus, the problem of establishing if $s \models_{\text{HBS}} L$ has data complexity P. Additionally, it is shown in App-12
[Alferes and Pereira 1996] that on normal logic programs, WFS and WFSX have the same conclusions. Let $P^m$ be a normal logic program and $L^m \in \text{HBP} \cup \lnot \text{HBP}$. It is stated in [Dantsin et al. 2001] that the problem of establishing if $L^m \in C_{WFSX}(P^m)$ is data complete for P. Therefore, the problem of establishing if $L' \in C_{WFSX}(P)$ is data complete for P. Now due to Proposition 7.4, it follows that the problem of establishing if $s \models_{WFSX} L$ is data complete for P.

Let $P'$ be the instantiated version of $P$. Note that $P'$ is exponentially larger than $P$ w.r.t. the program size. It holds that $L' \in C_{WFSX}(P)$ iff $L' \in C_{WFSX}(P')$. Therefore, the problem of establishing if $L' \in C_{WFSX}(P)$ has program complexity EXPTIME. Thus, the problem of establishing if $L^m \in C_{WFSX}(P^m)$ is program complete for EXPTIME. Therefore, the problem of establishing if $L' \in C_{WFSX}(P)$ is program complete for EXPTIME. Now due to Proposition 7.4, it follows that the problem of establishing if $s \models_{WFSX} L$ is program complete for EXPTIME. □

Proposition 7.3 Let $S$ be a modular rule base. Additionally, let $p \in \text{Pred}_P^S$ s.t. $\text{mode}_P^S(p) = \emptyset$, and let $L \in [p]_S \cup \lnot [p]_S$. It holds that: $s \models_{WFS} L$ iff $s \models_{WFSX} L$.

**Proof:** Note that $\Pi^d_{\text{AS},S}$ is a definite logic program. Thus, it holds that $C_{\text{AS}}(\Pi^d_{\text{AS},S}) = C_{WFSX}(\Pi^d_{S,S})$. Based on Corollary 6.3, it holds that: (i) $s \models_{\text{AS}} L$ iff $\tau^d_S(L) \in C_{\text{AS}}(\Pi^d_{S,S})$, and (ii) $s \models_{WFS} L$ iff $\tau^d_S(L) \in C_{WFSX}(\Pi^d_{S,S})$. Therefore, it follows that $s \models_{WFSX} L$ iff $s \models_{WFS} L$. □

Proposition 7.4 Let $s$ be a rule base s.t. $\text{Pred}_P^S = \emptyset$ and for all $p \in \text{Pred}_P^S$, $\text{mode}_P^S(p) = n$. Let $S = \{s\}$, let $p \in \text{Pred}_P^S$, and let $L \in [p]_S \cup \lnot [p]_S$. It holds that: (i) $s \models_{\text{AS}} L$ iff $L \in C_{\text{AS}}(P_s)$, and (ii) $s \models_{WFS} L$ iff $L \in C_{WFSX}(P_s)$.

**Proof:**
(i) Note that $\Pi^d_{S,S}$ results from $P_s$, if each $L' \in \text{HBP}$ is replaced by $\tau^d_S(L')$. Based on this and Corollary 6.3, it follows that: $s \models_{\text{AS}} L$ iff $\tau^d_S(L) \in C_{\text{AS}}(\Pi^d_{S,S})$ iff $L \in C_{\text{AS}}(P_s)$.
(ii) Similarly to (i), it follows that: $s \models_{WFS} L$ iff $\tau^d_S(L) \in C_{WFSX}(\Pi^d_{S,S})$ iff $L \in C_{WFSX}(P_s)$. □

Proposition 7.5 Let $S$ and $S'$ be modular rule bases s.t. $S \subseteq S'$ and for all $s \in S$ and $p \in \text{Pred}_P^S$, $\text{Import}_{S}^S(p) = \text{Import}_{S'}^{S'}(p)$. Let $L \in \text{HBS} \cup \lnot \text{HBS}$. It holds that: $s \models_{\text{BES}} L$ iff $s \models_{\text{BES}} L$, for $\text{SEM} \in \{\text{mAS, mWFS}\}$.

**Proof:**

**Case SEM = mAS:** Let $\text{pred}(L) = p$. If $p \in \text{Pred}_P^S$ and $\text{qual}(L)$ is undefined then let $m = \text{mode}_P^S(p)$. If $p \in \text{Pred}_P^S \setminus \text{Pred}_P^S$ and $\text{qual}(L)$ is undefined then let $m = \text{mode}_P^S(p)$. If $p \in \text{Pred}_P^S$ and $\text{qual}(L)$ is defined then let $m = \text{mode}_P^S(p)$.

Note that $\Pi^d_{S,S} = \Pi^d_{S',S'}$. Now, let $s \models_{\text{AS}} L$. Then, it follows from Theorem 6.1 that $\tau^d_S(L)$ is true according to all answer sets of $\Pi^d_{S,S}$. Thus, $\tau^d_S(L)$ is true according to all answer sets of $\Pi^d_{S,S'}$. Therefore, it follows from Theorem 6.1 that $s \models_{\text{AS}} L$. Similarly, it follows that if $s \models_{\text{AS}} L$ then $s \models_{\text{AS}} L$.

**Case SEM = mWFS:** Let $m$ be defined as in the previous case. Note that $\Pi^d_{S,S} = \Pi^d_{S',S'}$.

Now, let $s \models_{\text{WFS}} L$. Then, it follows from Theorem 6.1 that $\tau^d_S(L)$ is true according to all partial stable models of $\Pi^d_{S,S}$. Thus, $\tau^d_S(L)$ is true according to all partial

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stable models of $\Pi_{s,S'}$. Therefore, it follows from Theorem 6.1 that $s \models_{S'} L$. Similarly, it follows that if $s \models_{S'} L$ then $s \models_{S''} L$. □

**Proposition 7.7** Let $S$ and $S'$ be modular rule bases such that for all $s \in S$, there exists $s' \in S'$.

1. $\text{Nam}_s = \text{Nam}_{s'}$, $P_s \subseteq P_{s'}$, $\text{Pred}_s^0 \subseteq \text{Pred}_{s'}^0$, $\text{Pred}_s^2 \subseteq \text{Pred}_{s'}^2$.
2. For all $p \in \text{Pred}^0_s$:
   \[\text{scope}_s(p) \subseteq \text{scope}_{s'}(p), \quad \text{mode}_s^0(p) = \text{mode}_{s'}^0(p), \quad \text{context}_s(p) = \text{context}_{s'}(p),\]
   \[\text{Export}^0_s(p) \subseteq \text{Export}_{s'}^0(p),\]
3. For all $p \in \text{Pred}^2_s$:
   \[\text{mode}_s^1(p) = \text{mode}_{s'}^1(p) \text{ and } \text{Import}^2_s(p) \subseteq \text{Import}_{s'}^2(p)\]

Let $s \in S$ and $s' \in S'$ s.t. $\text{Nam}_s = \text{Nam}_{s'}$. Let $p \in \text{Pred}^0_s$ s.t. $\text{mode}_s^0(p) \in \{d,o\}$ and let $L \in [p]_S$. It holds that: if $s \models_{S'} L$ then $s' \models_{S'} L$, for $\text{SEM} \in \{\text{mAS, mWFS}\}$.

**Proof:**

**Case mode\(_s^0(p) = d\) and SEM = mAS:** Let $\Pi = \Pi_{s,S}$ and let $\Pi' = \Pi_{s',S'}$. Note that both $\Pi$ and $\Pi'$ are definite logic programs and that it holds $\Pi \subseteq \Pi'$. Additionally, note that the AS semantics of a definite logic program $P$ coincide with $\text{least}_{s'}^s(P)$.

Assume that $s \models_{S'} L$. Then, for every normal answer set of $s$ in reasoning mode $d$ w.r.t. $S, M$, it holds that $M_s^d(L) = 1$. Then, according to Theorem 6.1, for every answer set $M$ of $\Pi$, it holds that $M_s^d(L) \subseteq M$. Thus, $\tau_{s'}^s(L) \in \text{least}_{s'}^s(\Pi)$. It now follows that $\tau_{s'}^s(L) \subseteq \text{least}_{s'}^s(\Pi')$. Therefore, for every answer set $N$ of $\Pi'$, it holds that $\tau_{s'}^s(L) \subseteq N$. Thus, from Theorem 6.1, for every normal answer set of $s'$ in reasoning mode $d$ w.r.t. $S'$, $N$, it holds that $N_{s'}^d(L) = 1$. Thus, $s' \models_{S'} L$.

**Case mode\(_s^0(p) = d\) and SEM = mWFS:** Let $\Pi = \Pi_{s,S}$ and let $\Pi' = \Pi_{s',S'}$. Note that both $\Pi$ and $\Pi'$ are definite logic programs and that it holds $\Pi \subseteq \Pi'$. Additionally, note that the WFS semantics of a definite logic program $P$ coincide with $\text{least}_{s'}^s(P)$.

Assume that $s \models_{S'} L$. Then, for every extended answer set of $s$ in reasoning mode $d$ w.r.t. $S, M$, it holds that $M_s^d(L) = 1$. Then, according to Theorem 6.1, for every partial stable model $M$ of $\Pi$, it holds that $M_s^d(L) \subseteq M$. Thus, $\tau_{s'}^s(L) \in \text{least}_{s'}^s(\Pi)$. It now follows that $\tau_{s'}^s(L) \subseteq \text{least}_{s'}^s(\Pi')$. Therefore, for every partial stable model $N$ of $\Pi'$, it holds that $\tau_{s'}^s(L) \subseteq N$. Thus, from Theorem 6.1, for every extended answer set of $s'$ in reasoning mode $d$ w.r.t. $S'$, $N$, it holds that $N_{s'}^d(L) = 1$. Thus, $s' \models_{S'} L$.

**Case mode\(_s^0(p) = o\) and SEM = mAS:** Let $\Pi = \Pi_{s,S}$, let $\Pi' = \Pi_{s',S'}$, and let $P$ be the set of definite rules in $\Pi_{s,S}$. Note that it holds $P \subseteq \Pi \subseteq \Pi'$. In the case that $s$ is contradictory in reasoning mode $o$ w.r.t. $S'$, it holds that $s' \models_{S'} L$. Assume now that $s$ is not contradictory in reasoning mode $o$ w.r.t. $S'$. Additionally, assume that $s \models_{S'} L$ and let $M$ be a normal answer set of $s'$ in reasoning mode $o$ w.r.t. $S'$. Note that $M$ is consistent. It follows from Theorem 6.1 that $N_M$ is a consistent answer set of $\Pi'$. Thus, $N_M = \text{least}_{s'}^s(\Pi'_{/\text{AS}}N_M)$. We will show that $M_{s'}^d(L) = 1$.

Let $I = \{\tau_s^0(p'(c_1, ..., c_k)) \mid (t, o) \in D_{s,S}, p' \in \text{Pred}^0_s, \text{mode}_s^0(p') \in \{o, c\}\}$, and

(i) $\tau_s^0(\text{ctx}(p'(c_1, ..., c_k)) \in \text{least}^s_{s'}(P)$, if $\text{ctx}(p')$ is defined, or (ii) $c_1, ..., c_k \in HU_S$, otherwise. Additionally, let $I' = (I \cap N_M) \cup (\neg I \cap N_M)$ and let $N = \text{least}^s_{s'}(\Pi'_{/\text{AS}}I')$. It holds that $N$ is consistent and $N = \text{least}^s_{s'}(\Pi'_{/\text{AS}}N)$. Thus, $N$ is an answer set of $\Pi$. Let $N$ be a normal interpretation of $s$ in reasoning mode $o$ w.r.t. $S$ s.t. $N_N = N$.
Then, it follows from Theorem 6.1 that \( N \) is a normal answer set of \( s \) in reasoning mode \( o \) w.r.t. \( S \). Since \( s \models_{\lambda}^{\lambda_S} L \), it holds that \( N_{\lambda}^o(L) = 1 \). Thus, \( N(\tau_o^s(L)) = 1 \). It holds that \( \text{least}_o^\lambda (\Pi/\text{mAS} N) \subseteq \text{least}_o^\lambda (\Pi/\text{mAS} M) \). Thus, \( N \subseteq N_M \). It follows from this that \( N_M(\tau_o^s(L)) = 1 \). Therefore, \( M_o^s(L) = 1 \). Thus, for every normal answer set of \( s' \) in reasoning mode \( o \) w.r.t. \( S' \), \( M \), it holds that \( M_o^s(L) = 1 \). Therefore, \( s' \models_{\lambda}^{\lambda_S} L \).

Case mode \( o \) and \( \text{SEM} = \text{mWFS} \): Let \( \Pi = \Pi_o^o \) and let \( \Pi' = \Pi_o^{o'} \). Then, it follows from Proposition 5.16 that \( s' \models_{\lambda}^{\lambda_S} L \). Assume now that \( s' \) is not contradictory in reasoning mode \( o \) w.r.t. \( S' \). Assume that \( s \models_{\lambda}^{\lambda_S} L \) and let \( M \) be an extended answer set of \( s' \) in reasoning mode \( o \) w.r.t. \( S' \). It follows from Proposition 5.18 that \( M \) is consistent. Thus, it follows from Theorem 6.1 that \( N_M \) is a consistent partial stable model of \( \Pi' \). Thus, \( N_M = \text{Coh}(\text{least}_o^\lambda(\Pi/\text{WFS} N)) \). We will show that \( M_o^s(L) = 1 \).

Let \( I = \{\tau_o^s(p'(c_1, \ldots, c_k)) \mid (t, o) \in D_o^s, p' \in \text{Pred}_o^s, \text{mode}_o^s(p') \in \{o, c\} \text{, and} \}

(i) \( \tau_o^s(\text{ctxt}(p'(c_1, \ldots, c_k))) \in \text{least}_o^\lambda(P) \), if \( \text{ctxt}(p') \) is defined, or (ii) \( c_1, \ldots, c_k \in \text{Bu}_o \), otherwise \}. Additionally, let \( I' = (I \cap N_M) \cup (\neg I \cap N_M) \cup (I \cap N_M) \cup (\neg I) \cap N_M \) and let \( N = \text{Coh}(\text{least}_o^\lambda(\Pi/\text{WFS} N')) \). It holds that \( N \) is consistent and \( N = \text{Coh}(\text{least}_o^\lambda(\Pi/\text{WFS} N)) \). Thus, \( N \) is a partial stable model of \( \Pi ' \). Let \( N \) be an extended interpretation of \( s \) in reasoning mode \( o \) w.r.t. \( S' \). Then, according to Theorem 6.1 that \( N \) is an extended answer set of \( s \) in reasoning mode \( o \) w.r.t. \( S \). Since \( s \models_{\lambda}^{\lambda_S} L \), it holds that \( N_{\lambda}^o(L) = 1 \). Thus, \( N(\tau_o^s(L)) = 1 \). It holds that \( \text{Coh}(\text{least}_o^\lambda(\Pi/\text{WFS} N)) \subseteq \text{Coh}(\text{least}_o^\lambda(\Pi/\text{WFS} N_M)) \). Thus, \( N \subseteq N_M \). It follows from this that \( N_M(\tau_o^s(L)) = 1 \). Therefore, \( M_o^s(L) = 1 \). Thus, for every extended answer set of \( s' \) in reasoning mode \( o \) w.r.t. \( S' \), \( M \), it holds that \( M_o^s(L) = 1 \). Therefore, \( s' \models_{\lambda}^{\lambda_S} L \). □

**Proposition 7.8** Let \( S \) and \( S' \) be modular rule bases such that for all \( s \in S \), there exists \( s' \in S' \):

1. \( \text{Nam}_s = \text{Nam}_{s'}, \text{P}_s = \text{P}_{s'}, \text{Pred}_o^s = \text{Pred}_{o'}^s, \text{Pred}_o^p = \text{Pred}_{o'}^p, \) and
2. For all \( p \in \text{Pred}_o^s : \)
   - (a) \( \text{scope}_s(p) = \text{scope}_{s'}(p) \) and \( \text{Export}_o^s(p) = \text{Export}_{o'}^s(p) \),
   - (b) \( \text{mode}_o^s(p) \leq |	ext{mode}_o^s(p)| \),
   - (c) if \( (\text{mode}_o^s(p), |\text{mode}_o^s(p)|) \in \{o, c\} \) then \( \text{context}_o^s(p) = \text{context}_{o'}^s(p) \).
3. For all \( p \in \text{Pred}_o^p : \)
   - \( \text{mode}_o^p(p) \leq \text{mode}_o^p(p) \) and \( \text{Import}_o^p(p) \).

Let \( s \in S \) and \( s' \in S' \). Let \( \text{Nam}_s = \text{Nam}_{s'} \). Let \( p \in \text{Pred}_o^p \) s.t. \( \text{mode}_o^p(p) \in \{d, o\} \) and \( L \in [p]_S \) and \( \Pi = \Pi_o^o \) and let \( \Pi' = \Pi_o^{o'} \). It holds that: if \( s \models_{\lambda}^{\lambda_S} L \) then \( s' \models_{\lambda}^{\lambda_S} L \), for \( \text{SEM} \in \{\text{mAS}, \text{mWFS} \} \).

**Proof:**

**Case mode \( o \) and \( \text{SEM} = \text{mWFS} \):** Let \( \Pi = \Pi_o^o \) and let \( \Pi' = \Pi_o^{o'} \). Assume that \( s \models_{\lambda}^{\lambda_S} L \). Then, for every normal answer set of \( s \) in reasoning mode \( d \) w.r.t. \( S, M \), it holds that \( M_o^s(L) = 1 \). Then, according to Theorem 6.1, for every answer set \( M \) of \( \Pi \), it holds that \( \tau_o^s(L) \in M \). Therefore, for every answer set \( N \) of \( \Pi' \), it holds that \( \tau_o^s(L) \in N \). Thus, from Theorem 6.1, for every normal answer set of \( s' \) in reasoning mode \( d \) w.r.t. \( S', N \), it holds that
$N_d^a(L) = 1$. Thus, $s' \models_{S_d^a} L$.

Case mode $D_1$ is mode $D_2$, and $SEM = mWFS$: Let $\Pi = \Pi^a_{d,s}$ and let $\Pi' = \Pi^a_{s',S'}$. Note that it holds $\Pi = \Pi'$. Assume that $s \models_{S_d^a} L$. Then, for every extended answer set of $s$ in reasoning mode $d$ w.r.t. $S$, $M$, it holds that $M_s^a(L) = 1$. Then, according to Theorem 6.1, for every partial stable model $M$ of $\Pi$, it holds that $\tau_s^a(L) \subseteq M$. Therefore, for every partial stable model $N$ of $\Pi'$, it holds that $\tau_s^a(L) \notin N$. Thus, from Theorem 6.1, for every extended answer set of $s'$ in reasoning mode $d$ w.r.t. $S'$, $M$, it holds that $N_s^a(L) = 1$. Thus, $s' \models_{S_d^a} L$.

Case mode $D_2$ is mode $D_1$, and $SEM = mAS$: It is proved similarly to Case mode $D_1$ is mode $D_2$, and $SEM = mAS$: Let $\Pi = \Pi^a_{d,s}$ and let $\Pi' = \Pi^a_{s',S'}$. Note that $\Pi$ is a definite logic program. Let $P$ be the set of definite rules in $\Pi'$, and in the case that $s'$ is contradictory in reasoning mode $o$ w.r.t. $S'$, it holds that $s' \models_{S_d^a} L$. Assume now that $s'$ is not contradictory in reasoning mode $o$ w.r.t. $S'$. Additionally, assume that $s' \models_{S_d^a} L$ and let $M$ be a normal answer set of $s'$ in reasoning mode $o$ w.r.t. $S'$. Note that $M$ is consistent. Thus, it follows from Theorem 6.1 that $N_M$ is a consistent answer set of $\Pi'$. We will show that $M_s^a(L) = 1$.

Let $N = \min_{S_d^a}(\Pi)$, and in the case that $s'$ is contradictory in reasoning mode $o$ w.r.t. $S'$, it holds that $s' \models_{S_d^a} L$. Assume now that $s'$ is not contradictory in reasoning mode $o$ w.r.t. $S'$. Additionally, assume that $s' \models_{S_d^a} L$ and let $M$ be an extended answer set of $s'$ in reasoning mode $o$ w.r.t. $S'$. It follows from Proposition 5.18 that $M$ is consistent. Thus, it follows from Theorem 6.1 that $N_M$ is a consistent stable partial model of $\Pi'$. We will show that $M_s^a(L) = 1$.

Let $N = \min_{S_d^a}(\Pi)$, and in the case that $s'$ is contradictory in reasoning mode $o$ w.r.t. $S'$, it holds that $s' \models_{S_d^a} L$. Assume now that $s'$ is not contradictory in reasoning mode $o$ w.r.t. $S'$. Additionally, assume that $s' \models_{S_d^a} L$ and let $M$ be an extended interpretation of $s$ in reasoning mode $d$ w.r.t. $S$. Since $s \models_{S_d^a} L$, it holds that $N_s^a(L) = 1$.

Thus, $N_s^a(L) = 1$. It follows from this that $N_M(\tau_s^a(L)) = 1$. Thus, $M_s^a(L) = 1$. Therefore, $s' \models_{S_d^a} L$. App-16
Proposition 7.13. Let $S$ be a modular rule base and let $s \in S$. Let $p \in \text{Pred}^0_S$ s.t. $p$ is $c$-stratified in $s$ w.r.t. $S$ and let $L = p(c_1, ..., c_n)$, where $c_i \in \text{HU}_S$, for $i = 1, ..., n$. Let $\text{SEM} \in \{\text{mAS}, \text{mWFS}\}$.

(1) If $p$ is freely (positively or negatively) closed in $s$ then: $s \models^\text{SEM} L$ or $s \models^\text{SEM} \neg L$.

(2) If $p$ is (positively or negatively) closed in $s$ w.r.t. context $\text{ctxt}$ then:

\[ s \models^\text{SEM} L, \text{ or } s \models^\text{SEM} \neg L, \text{ or } s \models^\text{SEM} \sim \text{ctxt}(c_1, ..., c_n). \]

Proof:

1) Case $\text{SEM} = \text{mAS}$: First assume that $\text{mode}^\text{p}_p(p) = c^+$. Obviously, in the case that $s$ is contradictory in reasoning mode $c$ w.r.t. $S$ or $M^\text{mAS}_S(s) = \{\}$, statement 1. holds. Assume now that $s$ is not contradictory in reasoning mode $c$ w.r.t. $S$ and that $M^\text{mAS}_S(s) \neq \{\}$. We will denote $\Pi^{c-}_p$ by $\Pi$. Let $D$ be the smallest set s.t. (i) $\tau_c^-(\neg L) \in D$ and (ii) if there exists $r \in [\Pi]$ with $\text{Head}_r \in D$ then $\text{Body}^+_r \cup \text{Body}^-_r \subseteq D$. We define $\Pi_p = \{r \in [\Pi] | \text{Head}_r \in D\}$.

Let $A$ be an atom in $\text{HB}_{\Pi^p}$, we define: (i) $\tau(A) = A$, (ii) $\tau(\neg A) = \neg A$, (iii) $\tau(\neg \neg A) = \neg A$, and (iv) $\tau(\sim \neg A) = \sim \neg A$. Let $\Pi^{\text{NP}}_p$ be the normal logic program that results if we replace each rule $L_0 \leftarrow L_1, ..., L_n$ in $\Pi_p$ by $\tau(L_0) \leftarrow \tau(L_1), ..., \tau(L_n)$. It is easy to see that $\Pi^{\text{NP}}_p$ is a normal logic program. Since $p$ is $c$-stratified in $s$ w.r.t. $S$, it follows that $\Pi^{\text{NP}}_p$ is stratified. Thus, $\Pi^{\text{NP}}_p$ has a unique stable model. It follows from this that $\Pi_p$ has a unique answer set. Let $M \in M^\text{mAS}_S(s)$. Since $s$ is not contradictory in reasoning mode $c$ w.r.t. $S$, it follows that $M$ is consistent. Thus, it follows from Theorem 6.1 that $N_M$ is a consistent answer set of $\Pi$. Therefore, $N_M = \text{least}^N_\text{AS}(\Pi/\text{AS}(N_M \cap D))$. Due to (ii) in the definition of $D$, it holds that $N_M \cap D = \text{least}^N_\text{AS}(\Pi_p/\text{AS}(N_M \cap D))$. Now it follows from this that $N_M \cap D$ is the unique answer set of $\Pi_p$. Thus, for all $M \in M^\text{mAS}_S(s)$, $N_M(L(L)) = 1$ or for all $M \in M^\text{mAS}_S(s)$, $N_M(L(L)) = 1$. Note that in $\Pi_p$, there exist the rule $\sim \text{Nam}_i : c.p(c_1, ..., c_n) \leftarrow \sim \text{Nam}_i : c.p(c_1, ..., c_n)$. It follows from this that for all $M \in M^\text{mAS}_S(s)$, $N_M(L(L)) = 1$ or for all $M \in M^\text{mAS}_S(s)$, $M^\text{mAS}_S(L(L)) = 1$. Therefore, it follows that for all $M \in M^\text{mAS}_S(s)$, $M^\text{mAS}_S(L(L)) = 1$. Now statement 1. follows.

Now assume that $\text{mode}^\text{p}_p(p) = c^-$. The proof of statement 1. follows similarly to the previous case, with the difference that $D$ is defined as the smallest set s.t. (i) $\tau_c^-(\neg L) \in D$ and (ii) if there exists $r \in [\Pi]$ with $\text{Head}_r \in D$ then $\text{Body}^+_r \cup \text{Body}^-_r \subseteq D$. Additionally, note that $\Pi_p$ now contains the rule $\sim \text{Nam}_i : c.p(c_1, ..., c_n) \leftarrow \sim \text{Nam}_i : c.p(c_1, ..., c_n)$.

Case $\text{SEM} = \text{mWFS}$: First assume that $\text{mode}^\text{p}_p(p) = c^+$. Obviously, in the case that $M^\text{mWFS}_S(s) = \{M\}$, where $M$ is inconsistent or $M^\text{mWFS}_S(s) = \{\}$, statement 1. holds. Assume now that $M^\text{mWFS}_S(s) \neq \{M\}$, where $M$ is inconsistent and that $M^\text{mWFS}_S(s) \neq \{\}$. We will denote $\Pi^\text{mWFS}_p$ by $\Pi$. Let $D$ be the smallest set s.t. (i) $\tau_c^-(\neg L) \in D$ and (ii) if there exists $r \in [\Pi]$ with $\text{Head}_r \in D$ then $\text{Body}^+_r \cup \text{Body}^-_r \subseteq D$. We define $\Pi_p = \{r \in [\Pi] | \text{Head}_r \in D\}$.

Let $A$ be an atom in $\text{HB}_{\Pi^p}$, we define: (i) $\tau(A) = A$, (ii) $\tau(\neg A) = \neg A$, (iii) $\tau(\sim \neg A) = \sim A$, and (iv) $\tau(\sim \neg A) = \sim \neg A$. Let $\Pi^{\text{NP}}_p$ be the normal logic program that results if we replace each rule $L_0 \leftarrow L_1, ..., L_n$ in $\Pi_p$ by $\tau(L_0) \leftarrow \tau(L_1), ..., \tau(L_n)$. It is easy to see that $\Pi^{\text{NP}}_p$ is a normal logic program. Since $p$ is $c$-stratified in $s$ w.r.t. $S$, it follows that $\Pi^{\text{NP}}_p$ is stratified. Thus, $\Pi^{\text{NP}}_p$ has a
well-founded model $W^p$ s.t. for all $L' \in D$, $W^p(\tau(L')) = 1$ or $W^p(\tau(L')) = 0$.

We will denote $\text{mWFS}_{x,c}$ by $W$. Since $M_{c,t}(s) \neq \{M\}$, where $M$ is inconsistent, and $M_{c,t}(s) \neq \{\}$, it follows from Definition 5.21 that $W$ is consistent. Thus, it follows from Proposition 6.2 that $N_W$ is the well-founded model of $\Pi$ (according to $\text{WFSX}$ semantics) and that $N_W$ is consistent.

**Lemma:** It holds that $W^p = \{ \tau(A) \mid A \in N_W \cap (D \cup \sim D) \}$.

**Proof:** Consider the sequence $\{I_n\}_{n \in \mathbb{N}}$, which is defined recursively as follows: $I_0 = \{\}$ and $I_{n+1} = \text{least}^n(WFSX)I_n$. In [Alferes and Pereira 1996], it is shown that $I_n \subseteq I_{n+1}$, for $n \in \mathbb{N}$. Let $\lambda$ be the smallest ordinal s.t. $I_\lambda = I_{\lambda+1}$. In [Alferes and Pereira 1996], it is shown that $I_\lambda = W^p$.

Consider now the sequence $\{J_n\}_{n \in \mathbb{N}}$, which is defined recursively as follows: $J_0 = \{\}$ and $J_{n+1} = \text{Coh}(\text{least}(WFSX)J_n)$. Let $X$ be the smallest ordinal s.t. $J_X = J_{X+1}$. In [Alferes and Pereira 1996], it is shown that $J_X = N_W$.

To prove the Lemma, it is enough to show that $I_\alpha = \{ \tau(A) \mid A \in J_\alpha \cap (D \cup \sim D) \}$, for all $\alpha \in \mathbb{N}$. We will prove the statement by induction. For $\alpha = 0$, the statement holds. Assume now that the statement holds for $\alpha = k$, where $k \in \mathbb{N}$. We will show that it also holds for $\alpha = k+1$. Let $\sim L' \in J_k \cap D$. Due to the $\text{Coh}$ operator in the definition of $J_k$, it follows that $\sim L' \in J_k \cap \sim D$. Thus, $\tau(\sim L') \in I_k$. Therefore, $I_k(\tau(L')) = 0$. Since $I_k \subseteq I_{k+1}$, in follows that $I_{k+1}(\tau(L')) = 0$. Now, due to (ii) in the definition of $D$ and since $I_k = \{ \tau(A) \mid A \in J_k \cap (D \cup \sim D) \}$, it follows that $\text{least}^n(WFSX)I_k = \{ \tau(A) \mid A \in \text{least}^n(WFSX)J_k \cap (D \cup \sim D) \}$. Obviously, $\text{least}^n(WFSX)I_k \subseteq \{ \tau(A) \mid A \in \text{Coh}(\text{least}(WFSX)J_k) \cap (D \cup \sim D) \}$.

Let $L' \in \{ \tau(A) \mid A \in \text{Coh}(\text{least}(WFSX)J_k) \cap D \}$. Then, $I_k(\tau(A)) = 1$. $L' \subseteq \text{least}^n(WFSX)J_k \cap D$. Thus, $L' \subseteq \text{least}^n(WFSX)I_k$.

Now, let $L' \not\in \{ \tau(A) \mid A \in \text{Coh}(\text{least}(WFSX)J_k) \cap D \}$, because otherwise $N_W$ is inconsistent. Therefore, $L' \not\in \text{least}^n(WFSX)I_k$. Since for all $L' \in D$, $W^p(\tau(L')) = 1$ or $W^p(\tau(L')) = 0$, it follows that $\sim L' \not\in \text{least}^n(WFSX)I_k$. Therefore, it holds $\tau(A) \cap \text{Coh}(\text{least}(WFSX)J_k) \cap (D \cup \sim D) \not\subseteq \text{least}^n(WFSX)I_k$. Now it follows that $I_{k+1} = J_{k+1}$.

**End of Lemma**

Based on the fact that for all $L' \in D$, $W^p(\tau(L')) = 1$ or $W^p(\tau(L')) = 0$ and since $\Pi_r$ contains the rule $\sim \text{Nam}_s: c.p(c_1, ..., c_n) \rightarrow \sim \text{Nam}_s: c.p(c_1, ..., c_n)$, it follows that $W^p(\tau(L')) = 1$ or $W^p(\tau(\sim L')) = 1$. Now based on the Lemma, it follows that $N_W(\tau^*_s(L)) = 1$ or $N_W(\tau^*_s(\sim L)) = 1$. It follows from this that $W^p(\tau(L)) = 1$ or $W^p(\sim L) = 1$. Therefore, from Corollary 5.23, it follows statement 1.

Now assume that $\text{mode}^p(c) = c^+$. The proof of statement 2. follows similarly to the previous case, with the difference that $D$ is defined as the smallest set s.t. (i) $\tau^*_s(L) \in D$ and (ii) if there exists $r \in \Pi$ with $\text{Head}_r \subseteq D$ then $\text{Body}_r \cup \text{Body}_r \subseteq D$. Additionally, note that $\Pi_r$ now contains the rule $\sim \text{Nam}_s: c.p(c_1, ..., c_n) \rightarrow \sim \text{Nam}_s: c.p(c_1, ..., c_n)$.

2) Case $\text{SEM} = \text{mAS}$ and Case $\text{SEM} = \text{mWFS}$: These cases are proved as the corresponding cases in 1) but now $\Pi_r$ contains (i) the rule $\sim \text{Nam}_s: c.p(c_1, ..., c_n) \rightarrow \text{Nam}_s: c.p(c_1, ..., c_n)$, (ii) the rule $\sim \text{Nam}_s: c.p(c_1, ..., c_n) \rightarrow \sim \text{Nam}_s: c.p(c_1, ..., c_n)$, if $\text{mode}^p(c) = c^+$, and (ii) the rule $\sim \text{Nam}_s: c.p(c_1, ..., c_n) \rightarrow \sim \text{Nam}_s: c.p(c_1, ..., c_n)$.

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Nam:\cdot \cext(c_1, \ldots, c_n), \sim \neg Nam: \cdot p(c_1, \ldots, c_n), \text{if mode}^{p}(p) = c^- \quad \square

**Proposition 7.14** Let $S$ be a modular rule base and let $s \in S$ s.t. $M^{KS}_{c,S}(s) \neq \emptyset$. Additionally, let $p \in \text{Pred}_{c}^{S}$ s.t. $p$ is $c$-stratified in $S$ w.r.t. $S$, and let $L \in [p]_{c} \cup \sim [p]_{S}$. It holds that: $s \models^{S}_{c} L$ if $s \models^{SFS}_{c} L$.

**Proof:**

Case mode$^{p}_{c}(p) = c^{+}$: Let $L = p(c_1, \ldots, c_n)$ or $L = \neg p(c_1, \ldots, c_n)$ or $L = \sim \neg p(c_1, \ldots, c_n)$ or $L = \sim \neg \sim p(c_1, \ldots, c_n)$. In the case that $s$ is contradictory in reasoning mode $c$ w.r.t. $S$, it follows from Proposition 5.16 that $s \models^{S}_{c} L$ and $s \models^{SFS}_{c} L$. Assume now that $s$ is not contradictory in reasoning mode $c$ w.r.t. $S$. We will denote $M^{HS}_{c,S}$ by $P$. Let $\Delta$ be the smallest set s.t. (i) $\tau_{c}^{S} (\sim \neg p(c_1, \ldots, c_n)) \in D$ and (ii) if there exists $r \in [P]$ with $\text{Head}_{c} \in D$ then $\text{Body}_{c}^{+} \cup \text{Body}_{c}^{-} \subseteq D$. We define $P_{c} = \{ r \in [P] \mid \text{Head}_{c} \in D \}$.

Let $A$ be an atom in $\Pi_{P}$, we define: (i) $\tau(A) = A$, (ii) $\tau(\sim A) = \sim A$, (iii) $\tau(\neg A) = \neg A$, and (iv) $\tau(\sim \neg A) = \sim \neg A$. Let $\Pi^{HS}_{c}$ be the normal logic program that results, if we replace each rule $L_{0} \leftarrow L_{1}, \ldots, L_{n}$ in $P_{c}$ by $\tau(L_{0}) \leftarrow \tau(L_{1}), \ldots, \tau(L_{n})$. It is easy to see that $\Pi^{HS}_{c}$ is a normal logic program. Since $p$ is $c$-stratified in $S$ w.r.t. $S$, it follows that $\Pi^{HS}_{c}$ is stratified. Thus, $\Pi^{HS}_{c}$ has a unique stable model. It follows from this that $P_{c}$ has a unique answer set. Let $M \in M^{KS}_{c,S}(s)$. Since $s$ is not contradictory in reasoning mode $c$ w.r.t. $S$, it follows that $M$ is consistent. Thus, it follows, from Theorem 6.1, that $M$ is a consistent answer set of $P$. Therefore, $N_{M} = \text{least}_{c}^{\ast}(\Pi^{HS}_{c}, M)$. It follows, from the definition (ii) in the Definition of $D$, that $N_{M} \cap D = \text{least}_{c}^{\ast}(\Pi^{HS}_{c}, N_{M} \cap D)$). It follows from this that $N_{M} \cap D$ is the unique answer set of $P_{c}$. Additionally, $\Pi^{HS}_{c}$ has a well-founded model $W^{HS}_{c}$ and since $\Pi^{HS}_{c}$ is stratified, it holds (i) for all $L' \in D$, $W^{HS}_{c}(\tau(L')) = 1$ or $W^{HS}_{c}(\tau(L')) = 0$, and (ii) $W^{HS}_{c} = \{ \tau(A) \mid A \in (N_{M} \cap D) \cup \sim (D \sim N_{M}) \}$.

We will denote $\Pi^{SFS}_{c,S}$ by $W$. Since $M^{KS}_{c,S}(s) \neq \emptyset$, it follows from Theorem 6.1 that $P_{c}$ has an answer set. Thus, $P_{c}$ has a partial stable model. Now it follows from Theorem 6.1 that $M^{KS}_{c,S}(s) \neq \emptyset$. Thus, $W \in M^{KS}_{c,S}(s)$. Since $s$ is not contradictory in reasoning mode $c$ w.r.t. $S$, it follows from Proposition 5.16 that $W$ is consistent. Thus, it follows from Proposition 6.2 that $N_{W}$ is the well-founded model of $P_{c}$ (according to $WFSX$ semantics) and that $N_{W}$ is consistent. It follows from the Lemma in the proof of Proposition 7.13 that it holds $W^{HS}_{c} = \{ \tau(A) \mid A \in N_{W} \cap (D \cup \sim D) \}$. Thus, $(N_{M} \cap D) \cup \sim (D \sim N_{M}) = N_{W} \cap (D \sim N_{M})$.

Assume now that $s \models^{SFS}_{c} L$. Then, $M^{H}_{c}(L) = 1$. Thus, $\tau_{c}^{S}(L) \in (N_{M} \cap D) \cup \sim (D \sim N_{M})$. Therefore, $\tau_{c}^{S}(L) \in N_{W} \cap (D \cup \sim D)$. Thus, $W^{H}_{c}(L) = 1$. Since $\Pi^{SFS}_{c,S} = W$, it follows from Corollary 5.23 that $s \models^{SFS}_{c} L$.

Reversely, assume that $s \models^{SFS}_{c} L$. Since $\Pi^{SFS}_{c,S} = W$, it follows from Corollary 5.23 that $W^{H}_{c}(L) = 1$. Therefore, $\tau_{c}^{S}(L) \in N_{W} \cap (D \cup \sim D)$. Thus, $\tau_{c}^{S}(L) \in (N_{M} \cap D) \cup \sim (D \sim N_{M})$. Therefore, $M^{H}_{c}(L) = 1$. From this, it follows that for all $M \in M^{KS}_{c,S}(s)$, it holds that $M^{H}_{c}(L) = 1$. It now follows that $s \models^{SFS}_{c} L$.

Case mode$^{p}_{c}(p) = c^{-}$: The proof of this case follows similarly to the previous case with the difference that $D$ is defined as the smallest set s.t. (i) $\tau_{c}^{S} (p(c_1, \ldots, c_n)) \in D$ and (ii) if there exists $r \in [P]$ with $\text{Head}_{c} \in D$ then $\text{Body}_{c}^{+} \cup \text{Body}_{c}^{-} \subseteq D$. \quad \square