Deductive Inference for the Interiors and Exteriors of Horn Theories

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In this paper, we investigate deductive inference for interiors and exteriors of Horn knowledge bases, where interiors and exteriors were introduced by Makino and Ibaraki [Makino and Ibaraki 1996] to study stability properties of knowledge bases. We present a linear time algorithm for deduction for interiors and show that deduction is coNP-complete for exteriors. Under model-based representation, we show that the deduction problem for interiors is NP-complete while the one for exteriors is coNP-complete. As for Horn envelopes of exteriors, we show that it is linearly solvable under model-based representation, while it is coNP-complete under formula-based representation. We also discuss polynomially solvable cases for all the intractable problems.

1. INTRODUCTION

Knowledge-based systems are commonly used to store the sentences as our knowledge for the purpose of having automated reasoning such as deduction applied to them (see e.g., [Brachman and Levesque 2004]). Deductive inference is a fundamental mode of reasoning, and usually abstracted as follows: Given a knowledge base $KB$, assumed to capture our knowledge about the domain in question, and a query $\chi$ that is assumed to capture the situation at hand, decide whether $KB$ implies $\chi$, denoted by $KB \models \chi$, which can be understood as the question: “Is $\chi$ necessarily true given the current state of knowledge?”

In this paper, we consider interiors and exteriors of knowledge bases. Formally, for a given positive integer $\alpha$, $\alpha$-interior of $KB$, denoted by $\sigma^{-\alpha}(KB)$, is a knowledge that consists of the models (or assignments) $v$ satisfying that the $\alpha$-neighbors of $v$ are all models of $KB$, and $\alpha$-exterior of $KB$, denoted by $\sigma^{\alpha}(KB)$, is a knowledge that consists of the models $v$ satisfying that at least one of the $\alpha$-neighbors of $v$ is a model of.


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KB [Makino and Ibaraki 1996]. Intuitively, the interior consists of the models \( v \) that strongly satisfy \( KB \), since all neighbors of \( v \) are models of \( KB \), while the exterior consists of the models \( v \) that weakly satisfy \( KB \), since at least one of the \( \alpha \)-neighbors of \( v \) is a model of \( KB \). Here we note that \( v \) might not satisfy \( KB \), even if we say that it weakly satisfies \( KB \). As mentioned in [Makino and Ibaraki 1996], the interiors and exteriors of knowledge bases merit study in their own right, since they shed light on the structure of knowledge bases. Moreover, let us consider the situation in which knowledge base \( KB \) is not perfect in the sense that some sentences in \( KB \) are wrong and/or some are missing in \( KB \) (see also [Makino and Ibaraki 1996]).

Suppose that we use \( KB \) as a knowledge base for automated reasoning, say, deductive inference \( KB \models \chi \). Since \( KB \) does not represent real knowledge \( KB^* \), the reasoning result is no longer true. However, if we use the interior \( \sigma^{-\alpha}(KB) \) of \( KB \) as a knowledge base and have \( \sigma^{-\alpha}(KB) \not\models \chi \), then we can expect that the result is true for real knowledge \( KB^* \), since \( \sigma^{-\alpha}(KB) \) consists of models which strongly satisfy \( KB \). On the other hand, if we use the exterior \( \sigma^{\alpha}(KB) \) of \( KB \) as a knowledge base and have \( \sigma^{\alpha}(KB) \models \chi \), then we can expect that the result is true for real knowledge \( KB^* \), since \( \sigma^{\alpha}(KB) \) consists of models which weakly satisfy \( KB \). In this sense, the interiors and exteriors help to have safe reasoning.

In this paper, we restrict knowledge bases to be Horn. Note that Horn theories are ubiquitous in Computer Science, cf. [Makowsky 1987], and are of particular relevance in Artificial Intelligence and Databases. It is known that important reasoning problems like deductive inference and satisfiability checking, which are intractable for arbitrary propositional theories, are solvable in linear time for Horn theories (cf. [Dowling and Galliear 1983]). Because of these computational advantage of Horn theories, knowledge bases are sometimes approximated to Horn theories, even if the original knowledge base are not Horn [Kavvadias et al. 1993]. Thus it is important to study safe reasoning for Horn theories.

Main problems considered. In this paper, we study deductive inference for interiors and exteriors of propositional Horn theories. More precisely, we address the following problems:

- Given a Horn theory \( \Sigma \), a clause \( c \), and nonnegative integer \( \alpha \), we consider the problems of deciding if deductive queries hold for the \( \alpha \)-interior and exterior of \( \Sigma \), i.e., \( \sigma^{-\alpha}(\Sigma) \models c \) and \( \sigma^{\alpha}(\Sigma) \models c \). It is well-known [Dowling and Galliear 1983] that a deductive query for a Horn theory can be answered in linear time. Note that it is intractable to construct the interior and exterior for a Horn theory [Makino and Ibaraki 1996; Makino et al. 2003], and hence a direct method (i.e., first construct the interior (or exterior) and then check a deductive query) is not possible efficiently.
- We contrast traditional formula-based (syntactic) with model-based (semantic) representation of Horn theories. The latter form of representation has been proposed as an alternative form of representing and accessing a logical knowledge base, cf. [Dechter and Pearl 1992; Eiter et al. 1999; Eiter and Makino 2002; Kautz et al. 1993; 1995; Kavvadias et al. 1993; Khardon and Roth 1996; 1997]. In model-based reasoning, \( \Sigma \) is represented by a subset of its models \( \mathcal{M} \), which are commonly called characteristic models. As shown by Kautz et al. [Kautz et al. 1993], deductive inference can be done in polynomial time, given its characteristic models.
- Finally, we consider Horn approximations for the exteriors of Horn theories. Note that the interiors of Horn theories are Horn, while the exteriors might not be Horn. We deal with the least upper bounds, called the Horn envelopes [Selman and Kautz 1991], for the exteriors of Horn theories.
Under model-based representation, we show that the consistency problem and de-
As for Horn envelopes of exteriors of Horn theories, we show that it is linearly
We present a linear time algorithm for deduction for interiors of a given Horn theory,

Main results. We investigate the problems mentioned above from an algorithmical
viewpoint. For all the problems, we provide either polynomial time algorithms or
proofs of the intractability; thus, our work gives a complete picture of the tractabil-
ity/intractability frontier of deduction for interiors and exteriors of Horn theories. Our
main results can be summarized as follows (see Figure 1).

- We present a linear time algorithm for deduction for interiors of a given Horn theory, and show that it is coNP-complete for deduction for the exteriors. Thus, the positive
result for ordinary deduction for Horn theories extends to the interiors, but does not to
the exteriors. We also show that deduction for the exteriors is solvable in polynomial
time, if $\alpha$ is bounded by a constant or if $|N(c)|$ is bounded by a logarithm of the input
size, where $N(c)$ corresponds to the set of negative literals in $c$.
- Under model-based representation, we show that the consistency problem and de-
duction for interiors of Horn theories are both coNP-complete. As for exteriors, we
show that the deduction is coNP-complete. We also show that deduction for interiors is
solvable in polynomial time if $\alpha$ is bounded by a constant, and so is for the exteriors, if
$\alpha$ or $|P(c)|$ is bounded by a constant, or if $|N(c)|$ is bounded by a logarithm of the input
size, where $P(c)$ corresponds to the set of positive literals in $c$.
- As for Horn envelopes of exteriors of Horn theories, we show that it is linearly
solvable under model-based representation, while it is coNP-complete under formula-
based representation. The former contrasts to the negative result for the exteriors. We
also present a polynomial algorithm for formula-based representation, if $\alpha$ is bounded
by a constant or if $|N(c)|$ is bounded by a logarithm of the input size.

We remark that deduction for (envelopes of) exteriors for formula-based representa-
tions and exteriors for model-based representation is fixed-parameter tractable (FPT)
with respect to $|N(c)|$.

The rest of the paper is organized as follows. In the next section, we review the basic
concepts and fix notations. Sections 3 and 4 investigate deductive inference for the
interiors and exteriors of Horn theories. Section 5 considers deductive inference for
the envelopes of the exteriors of Horn theories.

2. PRELIMINARIES

Horn Theories
We assume a standard propositional language with atoms $At = \{x_1, x_2, \ldots, x_n\}$, where
each $x_i$ takes either value 1 (true) or 0 (false). A literal is either an atom $x_i$ or its
negation, which we denote by $\overline{x_i}$. The opposite of a literal $\ell$ is denoted by $\overline{\ell}$, and the
opposite of a set of literals $L$ by $\overline{L} = \{\ell \mid \ell \in L\}$. Furthermore, $\overline{Lit} = At \cup \overline{At}$ denotes the
set of all literals.

A clause is a disjunction $c = \bigvee_{i \in P(c)} x_i \vee \bigvee_{i \in N(c)} \overline{x_i}$ of literals, where $P(c)$ and $N(c)$
are the sets of indices whose corresponding variables occur positively and negatively
in $c$ and $P(c) \cap N(c) = \emptyset$. Dually, a term is conjunction $t = \bigwedge_{i \in P(t)} x_i \wedge \bigwedge_{i \in N(t)} \overline{x_i}$ of

\begin{tabular}{|c|c|c|}
\hline
 & Interiors & Exteriors & Envelopes of Exteriors \\
\hline
Formula-Based & P & coNP-complete* & coNP-complete* \\
\hline
Model-Based & NP-complete* & coNP-complete* & P \\
\hline
\end{tabular}

*: It is polynomially solvable, if $\alpha = O(1)$ or $|N(c)| = O(log |\Sigma|)$.
†: It is polynomially solvable, if $\alpha = O(1)$.
‡: It is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(log(n|char(\Sigma)|))$.

Fig. 1. Complexity of deduction for interiors and exteriors of Horn theories

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literals, where \( P(t) \) and \( N(t) \) are similarly defined. We also view clauses and terms as sets of literals. A conjunctive normal form (CNF) is a conjunction of clauses. A clause \( c \) is Horn, if \( |P(c)| \leq 1 \). A theory \( \Sigma \) is any set of formulas; it is Horn, if it is a set of Horn clauses. As usual, we identify \( \Sigma \) with \( \bigwedge_{c \in \Sigma} c \), and write \( c \in \varphi \), etc. It is known \cite{dowling1983} that the deductive problem for a Horn theory, i.e., deciding if \( \Sigma \models c \) for a clause \( c \) is solvable in linear time.

We recall that Horn theories have a well-known semantic characterization. An assignment is a vector \( v \in \{0,1\}^n \), whose \( i \)-th component is denoted by \( v_i \). For an assignment \( v \), let \( ON(v) = \{ i \mid v_i = 1 \} \) and \( OFF(v) = \{ i \mid v_i = 0 \} \). The value of a formula \( \varphi \) on an assignment \( v \), denoted \( \varphi(v) \), is inductively defined as usual; satisfaction of \( \varphi \) in \( v \), i.e., \( \varphi(v) = 1 \), will be denoted by \( v \models \varphi \). For a formula \( \varphi \) (resp., a theory \( \Sigma \)), an assignment \( v \) is called a model of \( \varphi \) (resp., \( \Sigma \)) if \( \varphi(v) = 1 \) (resp., \( \bigwedge_{c \in \Sigma} c(v) = 1 \)). The set of models of a formula \( \varphi \) (resp., theory \( \Sigma \)), denoted by \( \text{mod}(\varphi) \) (resp., \( \text{mod}(\Sigma) \)), and logical consequence \( \varphi \models \psi \) (resp., \( \Sigma \models \psi \)) are defined as usual. For two assignments \( v \) and \( w \), we denote by \( v \leq w \) the usual componentwise ordering, i.e., \( v_i \leq w_i \) for all \( i = 1, 2, \ldots, n \), where \( 0 \leq v_i \leq w_i \) means \( v_i \neq w_i \) and \( v \leq w \). Denote by \( v \wedge w \) componentwise AND of assignments \( v, w \in \{0,1\}^n \), and by \( \text{Cl}_n(\mathcal{M}) \) the closure of \( \mathcal{M} \subseteq \{0,1\}^n \) under \( \wedge \). Then, a theory \( \Sigma \) is Horn representable if and only if \( \text{mod}(\Sigma) = \text{Cl}_n(\text{mod}(\Sigma)) \) (see \cite{dechter1992,khardon1996} for proofs).

**Example 2.1.** Consider \( \mathcal{M}_1 = \{(0101), (0101), (1000)\} \) and \( \mathcal{M}_2 = \{(0101), (1001), (1000), (0001), (0000)\} \). Then, for \( v = (0101), w = (1000) \), we have \( v, w \in \mathcal{M}_1 \), while \( v \wedge w = (0000) \notin \mathcal{M}_1 \); hence \( \mathcal{M}_1 \) is not the set of models of a Horn theory. On the other hand, \( \text{Cl}_n(\mathcal{M}_2) = \mathcal{M}_2 \), thus \( \mathcal{M}_2 = \text{mod}(\mathcal{M}_2) \) for some Horn theory \( \Sigma_2 \).

As discussed by Kautz et al. \cite{kautz1993}, a Horn theory \( \Sigma \) is semantically represented by its characteristic models, where \( v \in \text{mod}(\Sigma) \) is called characteristic (or extreme \cite{dechter1992}), if \( v \notin \text{Cl}_n(\text{mod}(\Sigma) \setminus \{v\}) \). The set of all such models, the characteristic set of \( \Sigma \), is denoted by \( \text{char}(\Sigma) \). Note that \( \text{char}(\Sigma) \) is unique. E.g., \( (0101) \in \text{char}(\Sigma_2) \), while \( (0000) \notin \text{char}(\Sigma_2) \); we have \( \text{char}(\Sigma_2) = \mathcal{M}_1 \). It is known \cite{kautz1993} that the deductive query for a Horn theory \( \Sigma \) from the characteristic set \( \text{char}(\Sigma) \) can be done in linear time, i.e., \( O(n \text{char}(\Sigma)) \) time.

**Interior and Exterior of Theories**

For an assignment \( v \in \{0,1\}^n \) and a nonnegative integer \( \alpha \), its \( \alpha \)-neighborhood is defined by

\[
N_\alpha(v) = \{ w \in \{0,1\}^n \mid |ON(w) \triangle ON(v)| \leq \alpha \},
\]

where \( \triangle \) denotes the symmetric difference operator. Note that \( |ON(w) \triangle ON(v)| \) denotes the Hamming distance between \( w \) and \( v \), and \( |N_\alpha(v)| = \sum_{i=0}^{\alpha} \binom{n}{i} = O(n^{\alpha+1}) \). For a theory \( \Sigma \) and a nonnegative integer \( \alpha \), its \( \alpha \)-interior and \( \alpha \)-exterior of \( \Sigma \), denoted by \( \sigma_\alpha(\Sigma) \) and \( \sigma_\alpha(\Sigma) \) respectively, are theories defined by

\[
\begin{align*}
\text{mod}(\sigma_\alpha(\Sigma)) &= \{ v \in \{0,1\}^n \mid N_\alpha(v) \subseteq \text{mod}(\Sigma) \} \\
\text{mod}(\sigma_\alpha(\Sigma)) &= \{ v \in \{0,1\}^n \mid N_\alpha(v) \cap \text{mod}(\Sigma) \neq \emptyset \}.
\end{align*}
\]

By definition, \( \sigma_0(\Sigma) = \Sigma \), \( \sigma_\alpha(\Sigma) \models \sigma_\beta(\Sigma) \) for integers \( \alpha \) and \( \beta \) with \( \alpha < \beta \), and \( \sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\Sigma_2) \) holds for any integer \( \alpha \), if two theories \( \Sigma_1 \) and \( \Sigma_2 \) satisfy \( \Sigma_1 \models \Sigma_2 \).

**Example 2.2.** Let us consider a Horn theory \( \Sigma = \{ \overline{x}_1 \lor x_3, \overline{x}_2 \lor x_3, \overline{x}_2 \lor x_4 \} \) of 4 variables, where \( \text{mod}(\Sigma) \) is given by

\[
\text{mod}(\Sigma) = \{(1111), (1011), (1010), (0111), (0011), (0010), (0001), (0000)\}
\]
Then we have $\sigma_\alpha(\Sigma) = \{\emptyset\}$ for $\alpha \leq -2$, $\{x_1, x_2, x_3, x_4\}$ for $\alpha = -1$, $\Sigma$ for $\alpha = 0$, $\{x_1 \lor x_2 \lor x_3 \lor x_4\}$ for $\alpha = 1$, and $\emptyset$ for $\alpha \geq 2$. For example, (0011) is the unique model of $\text{mod}(\sigma_{-1}(\Sigma))$, since $N_1(0011) \subseteq \text{mod}(\Sigma)$ and $N_1(v) \not\subseteq \text{mod}(\Sigma)$ holds for all other assignments $v$. For the 1-exterior, we can see that all assignments $v$ with $(x_1 \lor x_2 \lor x_3 \lor x_4)(v) = 1$ satisfy $N_1(v) \cap \text{mod}(\Sigma) \neq \emptyset$, and no other such assignment exists. For example, (0101) is a model of $\sigma_1(\Sigma)$, since $(0111) \in N_1(0101) \cap \text{mod}(\Sigma)$. On the other hand, (1100) is not a model of $\sigma_1(\Sigma)$, since $N_1(1100) \cap \text{mod}(\Sigma) = \emptyset$. Notice that $\sigma_{-1}(\Sigma)$ is Horn, while $\sigma_1(\Sigma)$ is not.

Makino and Ibaraki [Makino and Ibaraki 1996] introduced the interiors and exteriors to analyze stability of Boolean functions, and studied their basic properties and complexity issues on them (see also [Makino et al. 2003]). For example, it is known [Makino and Ibaraki 1996] that, for a theory $\Sigma$ and nonnegative integers $\alpha$ and $\beta$, $\sigma_{-\alpha}(\sigma_{-\beta}(\Sigma)) = \sigma_{-\alpha-\beta}(\Sigma)$, $\sigma_\alpha(\sigma_\beta(\Sigma)) = \sigma_{\alpha+\beta}(\Sigma)$, and

$$\sigma_\alpha(\sigma_{-\beta}(\Sigma)) \models \sigma_{-\alpha}(\Sigma) \models \sigma_{-\beta}(\sigma_\alpha(\Sigma)).$$  \hfill (3)

For a nonnegative integer $\alpha$ and two theories $\Sigma_1$ and $\Sigma_2$, we have

$$\sigma_{-\alpha}(\Sigma_1 \cup \Sigma_2) = \sigma_{-\alpha}(\Sigma_1) \cup \sigma_{-\alpha}(\Sigma_2)$$ \hfill (4)

$$\sigma_\alpha(\Sigma_1 \cup \Sigma_2) \models \sigma_\alpha(\Sigma_1) \cup \sigma_\alpha(\Sigma_2),$$ \hfill (5)

where $\sigma_\alpha(\Sigma_1 \cup \Sigma_2) \neq \sigma_\alpha(\Sigma_1) \cup \sigma_\alpha(\Sigma_2)$ holds in general.

As demonstrated in Example 2.2, it is not difficult to see that interiors of any Horn theory are Horn, which is, for example, proved by (4) and Lemma 3.1, while the exteriors might be not Horn.

3. DEDUCTIVE INFERENCE FROM HORN THEORIES

In this section, we investigate deductive inference for the interiors and exteriors of a given Horn theory.
3.1. Interiors

Let us first consider deduction for $\alpha$-interiors of a Horn theory: Given a Horn theory $\Sigma$, a clause $c$, and a positive integer $\alpha$, decide if $\sigma_{-\alpha}(\Sigma) \models c$ holds. We show that the problem is solvable in linear time after showing a series of lemmas.

The following lemma is a basic property of interiors of a theory, where we regard $c$ as a set of literals.

**Lemma 3.1.** Let $c$ be a clause. Then for a nonnegative integer $\alpha$, we have $\sigma_{-\alpha}(c) = \bigvee_{|S| = \alpha + 1} (\bigwedge_{t \in S} t)$ if $\alpha < |c|$, and 0 (i.e., always false), otherwise.

For example, let us consider $c = x_1 \lor x_2 \lor \overline{x}_3 \lor \overline{x}_4$, $\alpha = 2$. Then we have $\sigma_{-\alpha}(c) = x_1 x_2 \overline{x}_3 \lor x_1 x_2 \overline{x}_4 \lor x_2 \overline{x}_3 \overline{x}_4 = (x_1 \lor x_2)(x_1 \lor \overline{x}_3)(x_1 \lor \overline{x}_4)(x_2 \lor \overline{x}_3)(x_2 \lor \overline{x}_4)(\overline{x}_3 \lor \overline{x}_4).

This lemma, together with (4), implies that for a CNF $\varphi$ and a nonnegative integer $\alpha$, we have

$$\sigma_{-\alpha}(\varphi) = \bigwedge_{c \in \varphi} \left( \bigvee_{S \subseteq c} \left( \bigwedge_{t \in S} t \right) \right) = \bigwedge_{c \in \varphi} \left( \bigwedge_{S \subseteq c} \left( \bigvee_{t \in S} t \right) \right),$$

if all $c \in \varphi$ satisfy $\alpha < |c|$, and 0 (i.e., always false), otherwise.

**Lemma 3.2.** Let $\Sigma$ be a Horn theory, and let $c$ be a clause. For a nonnegative integer $\alpha$, if there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N(c)| \leq \alpha - 1$ or $|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c)$, then we have $\sigma_{-\alpha}(\Sigma) = c$.

**Proof.** If $\Sigma$ has a clause $d$ such that $|N(d) \setminus N(c)| \leq \alpha - 1$, then $|N(d) \setminus N(c)| \cup P(d) \leq \alpha$ holds. Thus by Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i \in N(c) \cap N(d)} \overline{x}_i = c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) = c$.

On the other hand, if $\Sigma$ has a clause $d$ such that $|N(d) \setminus N(c)| = \alpha$ and $P(d) \subseteq P(c)$, then by Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i \in P(d)} x_i \lor \bigvee_{i \in N(c) \cap N(d)} x_i = c$. Therefore, by (4), $\sigma_{-\alpha}(\Sigma) = c$.

**Lemma 3.3.** Let $\Sigma$ be a Horn theory, and let $c$ be a clause. For a nonnegative integer $\alpha$, if (i) $|N(d) \setminus N(c)| \geq \alpha$ holds for all $d \in \Sigma$ and (ii) $\emptyset \not\subseteq P(d) \subseteq N(c)$ holds for all $d \in \Sigma$ with $|N(d) \setminus N(c)| = \alpha$, then we have $\sigma_{-\alpha}(\Sigma) \not= c$.

**Proof.** Let $v$ be the unique minimal assignment that does not satisfy $c$, i.e., $v_i = 1$ if $x_i \in c$ and 0, otherwise. We show that $v = \sigma_{-\alpha}(\Sigma)$, which implies $\sigma_{-\alpha}(\Sigma) \not= c$.

Let $d$ be a clause in $\Sigma$ with $|N(d) \setminus N(c)| \geq \alpha + 1$, and let $t$ be a term obtained by conjuncting arbitrary $\alpha + 1$ literals in $N(d) \setminus N(c)$. Then we have $t(v) = 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 3.1. On the other hand, for a clause $d$ in $\Sigma$ with $|N(d) \setminus N(c)| = \alpha$, let $t$ be a term obtained by conjuncting all literals in $(N(d) \setminus N(c)) \cup P(d)$. Then we have $|t| = \alpha + 1$ and $t \models \sigma_{-\alpha}(d)$ by Lemma 3.1. Moreover, it holds that $t(v) = 1$ by $P(d) \subseteq N(c)$. Therefore, by (4), we have $v \models \sigma_{-\alpha}(\Sigma)$.

By Lemmas 3.2 and 3.3, we can easily answer the deductive queries, if $\Sigma$ satisfies certain conditions mentioned in them. In the remaining case, we have the following lemma.

**Lemma 3.4.** For a Horn theory $\Sigma$ that satisfies none of the conditions in Lemmas 3.2 and 3.3, let $d$ be a clause in $\Sigma$ such that $|N(d) \setminus N(c)| = \alpha$, and $P(d) = P(d) \setminus (P(c) \cup N(c)) = \{\}$. Then $\sigma_{-\alpha}(\Sigma) \models c \lor x_j$ holds.

**Proof.** By Lemma 3.1, we have $\sigma_{-\alpha}(d) \models \bigvee_{i \in N(c) \cap N(d)} x_i \lor x_j = c \lor x_j$. This implies $\sigma_{-\alpha}(\Sigma) \models c \lor x_j$ by (4).
From this lemma, we have only to check a deductive query $\sigma_{-\alpha}(\Sigma) \models c \lor \tau_j$, instead of $\sigma_{-\alpha}(\Sigma) \models c$. Since $|c| < |c \lor \tau_j| \leq n$, we can answer the deduction by checking the conditions in Lemmas 3.2 and 3.3 at most $n$ times. Formally, this procedure is described as Algorithm 1 below.

Algorithm 1 \textsc{Deduction-Interior-from-Horn-Theory}

\textbf{Input}: A Horn theory $\Sigma$, a clause $c$ and a nonnegative integer $\alpha$.

\textbf{Output}: Yes, if $\sigma_{-\alpha}(\Sigma) \models c$; Otherwise, No.

\begin{enumerate}
\item \textbf{Step 0.} Let $N := N(c)$ and $P := P(c)$.
\item \textbf{Step 1.} /* Check the condition in Lemma 3.2. */
\begin{enumerate}
\item If there exists a clause $d \in \Sigma$ such that $|N(d) \setminus N| \leq \alpha - 1$ or $(|N(d) \setminus N| = \alpha$ and $P(d) \subseteq P)$, then output Yes and halt.
\end{enumerate}
\item \textbf{Step 2.} /* Check the condition in Lemma 3.3. */
\begin{enumerate}
\item If $P(d) \subseteq N$ holds for all $d \in \Sigma$ with $|N(d) \setminus N| = \alpha$, then output No and halt.
\end{enumerate}
\item \textbf{Step 3.} /* Update $N$ by Lemma 3.4. */
\begin{enumerate}
\item For a clause $d$ in $\Sigma$ such that $|N(d) \setminus N| = \alpha$ and $P(d) = P(d) \setminus (P \cup N) = \{j\}$, update $N := N \cup \{j\}$ and return to Step 1. \hfill $\Box$
\end{enumerate}
\end{enumerate}

We can see that a straightforward implementation of the algorithm requires $O(n(\|\Sigma\| + |c|))$ time, where $\|\Sigma\|$ denotes the length of $\Sigma$, i.e., $\|\Sigma\| = \sum_{d \in \Sigma} |d|$. The running time can be improved by properly maintaining $N(d) \setminus N$ for all $d \in \phi$. For each variable $x_i$, we prepare a pointer to clause $d$ with $i \in N(d)$. Then if $N$ is updated to $N := N \cup \{j\}$, then $N(d) \setminus N$ can also be updated in $O(1)$ time for each $d$ with $i \in N(d)$. This means that the total time to update $N(d) \setminus N$ for all $d \in \phi$ is linear in $\|\Sigma\|$. Moreover, if we additionally keep the size of $N(d) \setminus N$ for each $d \in \phi$, the conditions in Steps 1 and 2 can be checked in $O(1)$ time. Thus Algorithm 1 requires $O(\|\Sigma\| + |c|)$ time.

\textbf{Theorem 3.5.} The problem of deciding, given a Horn theory $\Sigma$, a clause $c$ and a nonnegative integer $\alpha$, whether $\sigma_{-\alpha}(\Sigma) \models c$ holds can be solved in linear time, i.e., $O(\|\Sigma\| + |c|)$ time.

\textbf{Example 3.6.} Let us consider a clause $c = x_1 \lor x_2 \lor \bar{x}_3 \lor x_4 \lor x_5$ and a Horn theory $\Sigma = \{d_1 = \bar{x}_1 \lor \bar{x}_3 \lor \bar{x}_4 \lor x_6, d_2 = \bar{x}_1 \lor x_2 \lor \bar{x}_3 \lor \bar{x}_4 \lor \bar{x}_6\}$. We apply Algorithm 1 to $c$, $\Sigma$, and $\alpha = 2$. Step 0 initializes $P = \{1, 2, 4, 5\}$ and $N = \{3\}$. In Step 1, we have $N(d_1) \setminus N = \{1, 4\}, P(d_1) \subseteq P$ and $N(d_2) \setminus N = \{1, 4, 6\}$; no clause $d$ in $\Sigma$ satisfies $|N(d) \setminus N| \leq \alpha - 1 (= 1)$ or $(|N(d) \setminus N| = \alpha (= 2)$ and $P(d) \subseteq P)$. In Step 2, $d_1$ satisfies $|N(d_1) \setminus N| = 2$ and $P(d_1) \subseteq N$. Step 3 updates $N := N \cup P(d_1) = \{3, 6\}$ and returns to Step 1. In Step 1, we have $N(d_2) \setminus N = \{1, 4\}$ (i.e., $|N(d_2) \setminus N| = 2$ and $P(d_2) = \{2\} \subseteq P$, and hence we output “Yes” and halt. This answer can be also verified by $\sigma_{-2}(\Sigma) = (\bar{x}_1 \lor \bar{x}_3)(\bar{x}_1 \lor \bar{x}_4)(\bar{x}_3 \lor \bar{x}_4)(\bar{x}_1 \lor x_2)(\bar{x}_3 \lor x_2)(\bar{x}_4 \lor x_2)(\bar{x}_1 \lor x_6)(\bar{x}_3 \lor x_6)(\bar{x}_4 \lor x_6)$, which is obtained by Lemma 3.1 and a few resolution steps.

\textbf{3.2. Exteriors}

Let us next consider deduction for $\alpha$-exteriors of a Horn theory. In contrast to the interior case, we have the following negative result.

\textbf{Theorem 3.7.} The problem of deciding, given a Horn theory $\Sigma$, a clause $c$ and a positive integer $\alpha$, whether $\sigma_{\alpha}(\Sigma) \models c$ holds is coNP-complete, even if $P(c) = \emptyset$ and $\Sigma$ is restricted to be both negative and bifunctive.
By definition, \( \sigma_\alpha(\Sigma) \not\models c \) if and only if there exists a model \( v \) of \( \Sigma \) such that some assignment in \( N_\alpha(v) \) does not satisfy \( c \). The latter is equivalent to the condition that there exists a model \( v \) of \( \Sigma \) such that \(| \text{ON}(v) \cap P(c) | + | \text{OFF}(v) \cap N(c) | \leq \alpha \), which can be checked in polynomial time. Thus the problem is in \( \text{coNP} \).

We then show the hardness by reducing a well-known \( \text{NP} \)-complete problem \( \text{INDEPENDENT SET} \) to the complement of our problem. \( \text{INDEPENDENT SET} \) is the problem of deciding if a given graph \( G = (V,E) \) has an independent set \( W \subseteq V \) such that \(|W| \geq k \) for a given integer \( k \). Here we call a subset \( W \subseteq V \) an independent set of \( G \) if \(|W \cap e| \leq 1 \) for all edges \( e \in E \). For a problem instance \( G = (V = \{1,2,\ldots,n \}, E) \) and \( k \) of \( \text{INDEPENDENT SET} \), let us define a Horn theory \( \Sigma_G \) over \( At = \{x_1,x_2,\ldots,x_n \} \) by

\[
\Sigma_G = \{(\bar{x}_i \lor \bar{x}_j) \mid \{i,j \} \in E \}.
\]

Let \( c = \lor_{i=1}^n \pi_i \) and \( \alpha = n - k \). Note that \((11\ldots 1)\) is the unique assignment that does not satisfy \( c \). Thus \( \sigma_\alpha(\Sigma) \not\models c \) if and only if \( \sigma_\alpha(\Sigma) \models (11\ldots 1) = 1 \). Since \( W \) is an independent set of \( G \) if and only if \( \Sigma_G \) contains a model \( w \) defined by \( \text{ON}(w) = W \), \( \sigma_\alpha(\Sigma_G) \models (11\ldots 1) = 1 \) is equivalent to the condition that \( G \) has an independent set of size at least \( k (= n - \alpha) \). This completes the proof. \( \square \)

We remark that this result can also be derived from the ones in [Makino and Ibaraki 1996].

However, by using the next lemma, a deductive query can be answered in polynomial time, if \( \alpha \) or \( N(c) \) is small.

**Lemma 3.8.** Let \( \Sigma_1 \) and \( \Sigma_2 \) be theories. For a nonnegative integer \( \alpha \), Then \( \sigma_\alpha(\Sigma_1) \models \Sigma_2 \) if and only if \( \Sigma_1 \models \sigma_{-\alpha}(\Sigma_2) \).

**Proof.** For the if part, if \( \Sigma_1 \models \sigma_{-\alpha}(\Sigma_2) \), then we have \( \sigma_\alpha(\Sigma_1) \models \sigma_\alpha(\sigma_{-\alpha}(\Sigma_2)) \models \Sigma_2 \) by (3). On the other hand, if \( \sigma_\alpha(\Sigma_1) \models \Sigma_2 \), then we have \( \Sigma_1 \models \sigma_{-\alpha}(\sigma_\alpha(\Sigma_1)) \models \sigma_{-\alpha}(\Sigma_2) \) by (3). \( \square \)

From Lemma 3.8, the deductive query for the \( \alpha \)-exterior of a theory \( \Sigma \), i.e., \( \sigma_\alpha(\Sigma) \models c \) for a given clause \( c \) is equivalent to the condition that \( \Sigma \models \sigma_{-\alpha}(c) \). Since we have \( \sigma_{-\alpha}(c) = \bigwedge_{\substack{S \subseteq N(c) \\ |S| \geq |N(c)| - \alpha}} \bigvee_{i \in S} c_i \bigvee \bigwedge_{i \in \overline{S}} x_i \bigvee x_j \) by Lemma 3.1, the deductive query for the \( \alpha \)-interior can be done by checking \((\binom{c}{\alpha})\) deductions for \( \Sigma \). More precisely, we have the following lemma.

**Lemma 3.9.** Let \( \Sigma \) be a Horn theory, let \( c \) be a clause, and \( \alpha \) be a nonnegative integer. Then \( \sigma_\alpha(\Sigma) \models c \) holds if and only if, for each subset \( S \) of \( N(c) \) such that \(|S| \geq |N(c)| - \alpha \), at least \( (\alpha - |N(c)| + |S|) + 1 \) \( j \)'s in \( P(c) \) satisfy \( \Sigma \models \bigvee_{i \in S} \pi_i \lor x_j \).

**Proof.** From Lemmas 3.1 and 3.8, \( \sigma_\alpha(\Sigma) \models c \) if and only if \( \Sigma \models \bigwedge_{\substack{d \in C(c) \\ |d| = \alpha}} d = \bigwedge_{\substack{S \subseteq N(c) \\ |S| \geq |N(c)| - \alpha}} \Phi_S \), where

\[
\Phi_S = \bigwedge_{\substack{S \subseteq N(c) \\ |S| \geq |N(c)| - \alpha}} d.
\]

It is known that for a Horn theory \( \Sigma \) and clause \( d \), \( \Sigma \models d \) if and only if \( \Sigma \models \bigvee_{i \in N(a)} \pi_i \lor x_i \) holds for some \( j \in P(d) \) (i.e., all the prime implicates of Horn theory are Horn). Therefore, for each \( S \subseteq N(c) \) with \(|S| \geq |N(c)| - \alpha \), we have \( \Phi_S \) if and only if there exists a set \( S' \subseteq P(c) \) such that \(|S'| = \alpha - |N(c)| + |S| + 1 \) and \( \Sigma \models \bigvee_{i \in S} \pi_i \lor x_j \) holds for all \( j \in S' \). This proves the lemma. \( \square \)

This lemma implies that the deductive query can be answered by checking the number of \( j \)'s in \( P(c) \) that satisfy \( \Sigma \models \bigvee_{i \in S} \pi_i \lor x_j \) for each \( S \). Since we can check this...
condition in linear time and there are \( \sum_{p=0}^{\alpha} \binom{|N(c)|}{p} \) such \( S \)'s, we have the following result, which complements Theorem 3.7 that the problem is intractable, even if \( P(c) = \emptyset \).

\[ \text{PROPOSITION 3.10. The problem of deciding, given a Horn theory } \Sigma, \text{ a clause } c \text{ and a nonnegative integer } \alpha, \text{ whether } \sigma_\alpha(\Sigma) \models c \text{ can be solved in } O\left( \sum_{p=0}^{\alpha} \binom{|N(c)|}{p} \right) \parallel \Sigma \parallel + |P(c)| \text{ time. In particular, it is polynomially solvable, if } \alpha = O(1) \text{ or } |N(c)| = O(\log \parallel \Sigma \parallel). \]

4. DEDUCTIVE INFERENCE FROM CHARACTERISTIC SETS

In this section, we consider the case when Horn knowledge bases are represented by characteristic sets. Contrary to formula-based representation, deductions for interiors and exteriors are both intractable, unless P=NP.

4.1. Interiors

We first present an algorithm to solve the deduction problem for the interiors of Horn theories. The algorithm requires exponential time in general, but it is polynomial when \( \alpha \) is small.

Let \( \Sigma \) be a Horn theory given by its characteristic set \( \text{char}(\Sigma) \), and let \( c \) be a clause. Then for a nonnegative integer \( \alpha \), we have

\[ \sigma_\alpha(\Sigma) \models c \text{ if and only if } \sigma_\alpha(\Sigma) \land \overline{c} \equiv 0. \tag{6} \]

Let \( v^* \) be the unique minimal assignment such that \( c(v^*) = 0 \) (i.e., \( \overline{c}(v^*) = 1 \)). By the definition of interiors, \( v^* \) is a model of \( \sigma_\alpha(\Sigma) \) if and only if all \( v \)'s in \( N_\alpha(v^*) \) are models of \( \Sigma \). Therefore, for each assignment \( v \) in \( N_\alpha(v^*) \), we check if \( v \in \text{mod}(\Sigma) \), which is equivalent to

\[ \bigwedge_{w \in \text{char}(\Sigma), w \supseteq v} w = v. \tag{7} \]

If (7) holds for all assignments \( v \) in \( N_\alpha(v^*) \), then we can immediately conclude by (6) that \( \sigma_\alpha(\Sigma) \models c \). On the other hand, if there exists an assignment \( v \) in \( N_\alpha(v^*) \) such that (7) does not hold, let \( J = ON(\bigwedge_{w \in \text{char}(\Sigma), w \supseteq v} w \setminus ON(v)) \). By definition, we have \( J \neq \emptyset \), and \( \Sigma \land \bigwedge_{i \in ON(v)} x_i \land \overline{x}_j \equiv 0 \) for all \( j \in J \), that is,

\[ \Sigma \models \bigvee_{i \in ON(v)} \overline{x}_i \lor x_j \text{ for all } j \in J. \tag{8} \]

If \( J \cap N(c) \neq \emptyset \), then by Lemma 3.1 and (8), we have \( \sigma_\alpha(\Sigma) \models \bigvee_{i \in ON(v) \cap N(c)} \overline{x}_i \), since \( |ON(v) \setminus N(c)| \leq \alpha - 1 \). This implies \( \sigma_\alpha(\Sigma) \models c \). On the other hand, if \( J \cap N(c) = \emptyset \), then by Lemma 3.1 and (8), we have

\[ \sigma_\alpha(\Sigma) \models \bigvee_{i \in ON(v) \cap N(c)} \overline{x}_i \lor x_j \equiv \bigvee_{i \in N(c)} \overline{x}_i \lor x_j \text{ for all } j \in J. \]

Thus, if \( J \) contains an index in \( P(c) \), then we can conclude that \( \sigma_\alpha(\Sigma) \models c \); Otherwise, we check the condition \( \sigma_\alpha(\Sigma) \models c \bigvee_{j \in J} \overline{x}_j \), instead of \( \sigma_\alpha(\Sigma) \models c \). Since a new clause \( d = c \lor \bigvee_{j \in J} \overline{x}_j \) is longer than \( c \), after at most \( n \) iterations, we can answer the deductive query. Formally, our algorithm can be described as Algorithm 2.

\[ \text{PROPOSITION 4.1. The problem of deciding, given the characteristic model } \text{char}(\Sigma) \text{ of a Horn theory } \Sigma, \text{ a clause } c \text{ and a nonnegative integer } \alpha, \text{ whether } \sigma_\alpha(\Sigma) \models c \text{ can be solved in } O(n^{\alpha+2} \text{char}(\Sigma)) \text{ time. In particular, it is polynomially solvable, if } \alpha = O(1). \]
The characteristic set

Yes, if \( \sigma_{-\alpha}(\Sigma) \models c \); Otherwise, No.

Step 0. Let \( N := N(c), d^{(1)} := c \) and \( q := 1 \).

Step 1. Let \( u \) be the unique minimal assignment such that \( d^{(q)}(u) = 0 \).

Step 2. For each \( v \) in \( N_a(u) \) do

If (7) does not hold,

then let \( v^{(q)} := v, J := ON(\bigwedge_{w \in char(\Sigma)} w) \setminus ON(v) \) and

\( q := q + 1 \).

If \( J \cap (N \cup P(c)) \neq \emptyset \), then output yes and halt.

Let \( N := N \cup J \) and \( d^{(q)} := \bigvee_{i \in N} x_i \vee \bigvee_{i \in P(c)} x_i \).

Go to Step 1.

end{for}

Step 3. Output No and halt. □

Proof. Since we can see algorithm DEDUCTION-INTERIOR-FROM-CHARSET correctly answers a deductive query from the discussion before the description, we only estimate the running time of the algorithm.

Steps 0, 1 and 3 require \( O(n) \) time. Step 2 requires \( O(n^{\alpha + 1}|char(\Sigma)|) \) time, since (7) can be checked in \( O(n|char(\Sigma)|) \) time. Since we have at most \( n \) iterations between Steps 1 and 2, the algorithm requires \( O(n^{\alpha + 2}|char(\Sigma)|) \) time. □

Example 4.2. Let us consider a Horn theory \( \Sigma \) with \( char(\Sigma) = \{(1111), (1011), (1010), (0111), (0011)\} \) given in Example 2.2 (see Figure 2), and let \( c_1 = x_1 \vee x_2 \vee x_3 \) and \( c_2 = x_1 \vee x_2 \vee \bar{x}_3 \). It is easy to see that \( \sigma_{-1}(\Sigma) \models c_1 \) and \( \sigma_{-1}(\Sigma) \not\models c_2 \) by \( \sigma_{-1}(\Sigma) = \bar{x}_1 \bar{x}_2 x_3 x_4 \). Here, we execute Algorithm 2 for \( c_1 \) and \( c_2 \). For \( c_1 \), let \( N = \{1\} \) and \( d^{(1)} = \bar{x}_1 \vee x_2 \vee x_3 \) in Step 0. Note that (1000) is the unique minimal assignment of \( d^{(1)} = 0 \). In Step 2, we see that \( v = (1100) \in N_1(1000) \) does not satisfy (7), because \( \bigvee_{w \in char(\Sigma)} w = (1111) \neq v \). By \( J = \{3, 4\} \), we have \( J \cap (N \cup P(c_1)) = \{3\} \neq \emptyset \), and hence the algorithm answers “Yes” and halts. For \( c_2 \), let \( N = \{3\} \) and \( d^{(1)} = x_1 \vee x_2 \vee \bar{x}_3 \) in Step 0. Note that (0010) is the unique minimal assignment of \( d^{(1)} = 0 \). In Step 2, we see that \( v = (0110) \in N_1(0010) \) does not satisfy (7), because \( \bigvee_{w \in char(\Sigma)} w = (1011) \neq v \).

By \( J = \{4\} \), we have \( J \cap (N \cup P(c)) = \emptyset \). Thus we update \( N \) to \( N = \{3, 4\} \) and let \( d^{(2)} = x_1 \vee x_2 \vee \bar{x}_3 \vee \bar{x}_4 \). Again, Step 1 computes the unique minimal assignment (0011) of \( d^{(2)} = 0 \). Then any vector in \( N_1(0011) \) satisfies (7) and hence we output “No” and halt.

However, in general, the problem is intractable, which contrasts with the formula-model representation.

Theorem 4.3. The problem of deciding, given the characteristic set \( char(\Sigma) \) of a Horn theory \( \Sigma \) and a positive integer \( \alpha \), whether \( \sigma_{-\alpha}(\Sigma) \) is consistent, i.e., \( mod(\sigma_{-\alpha}(\Sigma)) \neq \emptyset \), is coNP-complete.

Proof. Let us first show that the problem belongs to coNP. Apply Algorithm DEDUCTION-INTERIOR-FROM-CHARSET to the instance \((char(\Sigma), c = \emptyset, \alpha)\). If \( \sigma_{-\alpha}(\Sigma) \)
is not consistent, then the algorithm constructs a series of vectors, \(v^{(1)}, \ldots, v^{(k)}, k \leq n\), in Step 2. We can see that these vectors form a polynomial-size witness to the inconsistency of \(\sigma_{-\alpha}(\Sigma)\). In fact, if we are given these vectors, we can compute clauses \(d^{(1)}, d^{(2)}, \ldots, d^{(k)}\) and reduce the deduction problem \(\sigma_{-\alpha}(\Sigma) \models c = d^{(1)}\) to \(\sigma_{-\alpha}(\Sigma) \models d^{(2)}\), \(\sigma_{-\alpha}(\Sigma) \models d^{(3)}, \ldots, \sigma_{-\alpha}(\Sigma) \models d^{(k)},\) and conclude the inconsistency. Since all the computation can be done in polynomial time, the problem belongs to \(\text{coNP}\).

We show the \(\text{coNP}\)-hardness by reducing \(\text{INDEPENDENT SET}\) to our problem. Given a problem instance \(G = (V = \{1, 2, \ldots, n\}, E)\) and \(k\) of \(\text{INDEPENDENT SET}\), let us define a Horn theory \(\Sigma_G\) over \(At = \{x_1, x_2, \ldots, x_n\}\) by

\[
\text{char}(\Sigma_G) = \{v^{(i,j)}, v^{(i,j,l)} \mid \{i, j\} \in E, l \in V \setminus \{i, j\}\},
\]

where \(v^{(i,j)}\) and \(v^{(i,j,l)}\) are respectively the vectors defined by \(O\)FF\((v^{(i,j)}) = \{i, j\}\) and \(O\)FF\((v^{(i,j,l)}) = \{i, j, l\}\). Let \(\alpha = n - k\). Note that \(\Sigma_G\) is a negative theory, and hence \(\sigma_{-\alpha}(\Sigma_G)\) is consistent if and only if \((00 \cdots 0)\) is a model of \(\sigma_{-\alpha}(\Sigma_G)\). Moreover, the latter condition is equivalent to the one that \(G\) has no independent set of size at least \(k(= n - \alpha)\). This completes the proof. \(\square\)

This result immediately implies the following corollary.

**Corollary 4.4.** The problem of deciding, given the characteristic set \(\text{char}(\Sigma)\) of a Horn theory \(\Sigma\), a clause \(c\) and a positive integer \(\alpha\), whether \(\sigma_{-\alpha}(\Sigma) \models c\) holds is \(\text{NP}\)-complete, even if \(c = \emptyset\).

Note that, different from the other hardness results, the hardness does not require \(c\) to be large enough.

## 4.2. Exteriors

Let us consider the exteriors. Similarly to the formula-based representation, we have the following negative result.

**Theorem 4.5.** The problem of deciding, given the characteristic set \(\text{char}(\Sigma)\) of a Horn theory \(\Sigma\), a clause \(c\) and a positive integer \(\alpha\), whether \(\sigma_{\alpha}(\Sigma) \models c\) holds is \(\text{coNP}\)-complete.

**Proof.** From Lemmas 3.1 and 3.8, \(\sigma_{\alpha}(\Sigma) \not\models c\) if and only if there exists a subclause \(d\) of \(c\) such that \(|d| = |c| - \alpha\) and \(\Sigma \not\models d\). This \(d\) is a witness that the problem belongs to \(\text{coNP}\).

We then show the hardness by a reduction from \(\text{VERTEX COVER}\) which is known to be \(\text{NP}\)-hard. \(\text{VERTEX COVER}\) is the problem to decide if a given graph \(G = (V, E)\) has a vertex cover \(U\) such that \(|U| \leq k\) for a given integer \(k(< |V|)\). Here \(U \subseteq V\) is called a vertex cover if \(U \cap e \neq \emptyset\) holds for all \(e \in E\). For this problem instance, we construct our problem instance. For each \(e \in E\), let \(W_e = \{e_1, e_2, \ldots, e_{|V|}\}\), and let \(W = \bigcup_{e \in E} W_e\). Let \(m(v), v \in V\), be an assignment over \(V \cup W\) such that

\[
\text{ON}(m(v)) = (V \setminus \{v\}) \cup \bigcup_{v \in e} W_e,
\]

and let \(\text{char}(\Sigma)\) be the characteristic set for some Horn theory \(\Sigma\) defined by \(\text{char}(\Sigma) = \{m(v) \mid v \in V\}\). We define \(c\) and \(\alpha\) by

\[
c = \bigvee_{i \in V} \pi_i \lor \bigvee_{i \in W} x_i \quad \text{and} \quad \alpha = k,
\]

respectively. For this instance, we show that \(\sigma_{\alpha}(\Sigma) \not\models c\) if and only if the corresponding \(G\) has a vertex cover \(U\) of size at most \(k(= \alpha)\).

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Let $m^{(U)} \overset{\text{def}}{=} \bigwedge_{w \in U} m^{(w)}$, which is a model of $\Sigma$ by the intersection property of a Horn theory. Note that $m^{(U)}$ does not satisfy a clause $\bar{d} = \bigvee_{i \in V \setminus U} \bar{x}_i \lor \bigvee_{i \in W} x_i$. Since $d$ is a subclause of $c$ of length at least $|c| - \alpha$, $m^{(U)}$ is not a model of $\sigma_{-\alpha}(c)$ by Lemma 3.1. This completes the if part by Lemma 3.8.

For the only-if part, let us assume that $\sigma_{\alpha}(\Sigma) \not\models c$. Then by Lemmas 3.1 and 3.8, there exists a subclause $d$ of $c$ such that $|d| = |c| - \alpha$ and $\Sigma \not\models d$. This implies that $\Sigma \not\models \bar{d}$ contains a model $m$. By $\alpha < |V|$, for each $c \in E$, there exist an index $j$ in $W$ such that $m_j = 0$. Since any model $m'$ in $\Sigma$ satisfies either $m'_i = 0$ or $m'_i = 1$ for all $i \in W$, we have $m_i = 0$ for all $i \in W$. This means that $V \setminus ON(m)$ is a vertex cover of $G$, and since $|V \setminus ON(m)| \leq k$, we have the only-if part.

By using Lemma 3.9, we can see that the problem can be solved in polynomial time, if $\alpha$ or $|N(c)|$ is small. Namely, for each subset $S$ of $N(c)$ such that $|S| > |N(c)| - \alpha$, let $v^S$ denotes the assignment such that $ON(v^S) = S$. Then $w^S = \bigwedge_{w \in char(\Sigma)} w$ is the unique minimal model of $\Sigma$ such that $ON(w^S) \supseteq S$, and hence it follows from Lemma 3.9 that it is enough to check if $|ON(w^S) \cap P(c)| \geq \alpha - |N(c)| + |S| + 1$. Clearly, this can be done in in $O\left(\sum_{p=0}^{\alpha} \binom{|N(c)|}{p} |n| |char(\Sigma)|\right)$ time.

Moreover, if $|P(c)|$ is small, then the problem also becomes tractable, which contrasts with Theorem 3.7.

**Lemma 4.6.** Let $\Sigma$ be a theory, let $c$ be a clause, and $\alpha$ be a nonnegative integer. Then $\sigma_{\alpha}(\Sigma) \models c$ holds if and only if each $S \subseteq P(c)$ such that $|S| \geq |P(c)| - \alpha$ satisfies

$$|OFF(w) \cap N(c)| \geq \alpha - |P(c)| + |S| + 1 \quad (9)$$

for all models $w$ of $\Sigma$ such that $OFF(w) \cap P(c) = S$.

Note that the lemma holds for any theory $\Sigma$, and (9) is monotone in the sense that, if an assignment $w$ satisfies (9), then all assignments $v$ with $v < w$ also satisfy it. Thus it is sufficient to check if (9) holds for all maximal models $w$ of $\Sigma$ such that $OFF(w) \cap P(c) = S$. If $\Sigma$ is Horn, then such maximal models $w$ can be obtained from $w(i)$ ($i \in S$) with $i \in OFF(w(i)) \cap P(c) \subseteq S$ by their intersection $w = \bigwedge_{i \in S} w(i)$. Thus we can answer the deduction problem in $O\left(n \sum_{p=0}^{\alpha} \binom{|P(c)|}{p} |\text{char}(\Sigma)| \right)$ time.

**Proposition 4.7.** The problem of deciding, given the characteristic set $-char(\Sigma)$ of a Horn theory, a clause $c$ and a nonnegative integer $\alpha$, whether $\sigma_{\alpha}(\Sigma) \models c$ holds can be solved in $O\left(n \min\{\sum_{p=0}^{\alpha} \binom{|P(c)|}{p} |\text{char}(\Sigma)|, \sum_{p=0}^{\alpha} \binom{|P(c)|}{p} |\text{char}(\Sigma)|\right)$ time. In particular, it is polynomially solvable, if $\alpha = O(1)$, $|P(c)| = O(1)$, or $|N(c)| = O(\log(n|\text{char}(\Sigma)|))$.

5. DEDUCTIVE INFERENCE FOR ENVELOPES OF THE EXTERIORS OF HORN THEORIES

We have considered deduction for interiors and exteriors of Horn theories. As mentioned before, the interiors of Horn theories are also Horn, while this does not hold for the exteriors. This means that the exteriors of Horn theories might lose beneficial properties of Horn theories. One of the ways to overcome such a hurdle is Horn Approximation, that is, approximating a theory by a Horn theory [Selman and Kautz 1991]. There are several methods for approximation, but one of the most natural ones is to approximate a theory by its Horn envelope. For a theory $\Sigma$, its Horn envelope is the Horn theory $\Sigma_h$ such that mod($\Sigma_h$) = CL$_\lambda$(mod($\Sigma$)). Since Horn theories are closed under intersection, the Horn envelope is the least Horn upper bound for $\Sigma$, i.e., char($\Sigma_h$) $\geq$ char($\Sigma$)
and there exists no Horn theory $\Sigma^*$ such that $\text{char}(\Sigma_\alpha) \supseteq \text{char}(\Sigma^*) \supseteq \text{char}(\Sigma)$. In this section, we consider deduction for Horn envelopes of exteriors of Horn theories, i.e., $\sigma_\alpha(\Sigma_\alpha) \models c$.

5.1. Model-Based Representations

Let us first consider the case in which knowledge bases are represented by characteristic sets.

**Lemma 5.1.** Let $\Sigma$ be a Horn theory, and let $\alpha$ be a nonnegative integer. Then we have

$$\text{mod}(\sigma_\alpha(\Sigma_\alpha)) = \text{Cl}_\lambda(\bigcup_{v \in \text{char}(\Sigma)} \mathcal{N}_\alpha(v)).$$

**Proof.** By definition, $\text{mod}(\sigma_\alpha(\Sigma_\alpha)) = \text{Cl}_\lambda(\text{mod}(\sigma_\alpha(\Sigma))) \supseteq \text{Cl}_\lambda(\bigcup_{\nu \in \text{char}(\Sigma)} \mathcal{N}_\alpha(\nu))$ holds. For the converse direction, let $\nu^*$ be a model of Horn envelope of the $\alpha$-exterior, i.e., $\nu^* \in \text{mod}(\sigma_\alpha(\Sigma_\alpha))$. Then $\nu^*$ can be represented by $\nu^* = \bigwedge_{\nu \in W} \nu$ for some $W \subseteq \text{mod}(\sigma_\alpha(\Sigma))$. For $w \in W$, let $w$ be a model of $\Sigma$ such that $w$ is contained in $\mathcal{N}_\alpha(u)$. Since such a $\nu^*$ can be represented by $u = \bigwedge_{\nu \in S_u} \nu$, for some $S_u \subseteq \text{char}(\Sigma)$, $w$ is represented by $w = \bigwedge_{\nu \in S_u} \nu^*$, where $\nu^*$ is defined by $\nu^*_i = 0$ if $i \in ON(w) \setminus ON(u)$, 1 if $i \in ON(w) \setminus ON(u)$, and $\nu^*_i$ otherwise. Note that $|ON(w)\Delta ON(u)| \leq |ON(w)\Delta ON(u)| \leq \alpha$, that is, $\nu^* \in \mathcal{N}_\alpha(v)$, and $w$ belongs to $\text{Cl}_\lambda(\bigcup_{\nu \in S_u} \mathcal{N}_\alpha(\nu))$. This, together with $\nu^* = \bigwedge_{\nu \in W} \nu$, implies that $\nu^*$ also belongs to $\text{Cl}_\lambda(\bigcup_{\nu \in \text{char}(\Sigma)} \mathcal{N}_\alpha(\nu))$.

$\square$

For a clause $c$, let $\nu^*$ be the unique minimal assignment such that $c(\nu^*) = 0$. We recall that, for a Horn theory $\Phi$,

$$\Phi \models c \text{ if and only if } c(\bigwedge_{\nu \in \text{char}(\Phi)} \nu) = 1.$$  \hspace{1cm} (11)

Therefore, Lemma 5.1 immediately implies an algorithm for deduction for $\sigma_\alpha(\Sigma_\alpha)$ from $\text{char}(\Sigma)$, since we have $\text{char}(\sigma_\alpha(\Sigma_\alpha)) \subseteq \bigcup_{\nu \in \text{char}(\Sigma)} \mathcal{N}_\alpha(\nu) \subseteq \sigma_\alpha(\Sigma_\alpha)$. However, for a general $\alpha$, $\bigcup_{\nu \in \text{char}(\Sigma)} \mathcal{N}_\alpha(\nu)$ is exponentially larger than $\text{char}(\Sigma)$, and hence this direct method is not efficient. The following lemma helps developing a polynomial time algorithm.

**Lemma 5.2.** Let $\Sigma$ be a Horn theory, let $c$ be a clause, and let $\alpha$ be a nonnegative integer. Then $\sigma_\alpha(\Sigma_\alpha) \models c$ holds if and only if the following two conditions are satisfied.

(i). $|OFF(v) \cap N(c)| \geq \alpha$ holds for all $v \in \text{char}(\Sigma)$.

(ii). If $S = \{v \in \text{char}(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\} \neq \emptyset$, $P(c)$ is not covered with $OFF(v)$ for models $v$ in $S$, i.e., $P(c) \not\subseteq \bigcup_{v \in \text{char}(\Sigma)} OFF(v)$.

**Proof.** To show the if part, let us first assume that (i) and (ii) in the lemma holds. Let $v$ be a model in $\text{char}(\Sigma)$ such that $|OFF(v) \cap N(c)| \geq \alpha$. Then all assignments $w$ in $\mathcal{N}_\alpha(v)$ satisfy $OFF(w) \cap N(c) \neq \emptyset$. Therefore, if all the models $v$ in $\text{char}(\Sigma)$ satisfy $|OFF(v) \cap N(c)| > \alpha$, then by Lemma 5.1, we have $OFF(w) \cap N(c) \neq \emptyset$ for any model $w$ of $\sigma_\alpha(\Sigma_\alpha)$. This implies $\sigma_\alpha(\Sigma_\alpha) \models c$. Therefore, let us consider the case when $S = \{v \in \text{char}(\Sigma) \mid |OFF(v) \cap N(c)| = \alpha\}$ is not empty. Let $\nu^*$ be the unique minimal assignment such that $c(\nu^*) = 0$. Then by Lemma 5.1, we have

$$\{v \in \text{char}(\sigma_\alpha(\Sigma_\alpha)) \mid v \geq \nu^*\}$$
The problem of deciding, given a Horn theory \( \Sigma \), whether \( \sigma_\alpha(\Sigma)_c \models c \) holds can be solved in linear time.

We remark that this contrasts with Corollary 4.4. Namely, if we are given the characteristic set \( char(\Sigma) \) of a Horn theory \( \Sigma \), a clause \( c \) and a nonnegative integer \( \alpha \), whether \( \sigma_\alpha(\Sigma)_c \models c \) holds can be solved in polynomial time, while it is coNP-complete to decide if \( \sigma_\alpha(\Sigma) \models c \).

5.2. Formula-Based Representation

Recall that negative theories (i.e., theories consisting of clauses with no positive literal) are Horn and the exteriors of negative theories are also negative, and hence Horn. This means that, for a negative theory \( \Sigma \), we have \( \sigma_\alpha(\Sigma)_c = \sigma_\alpha(\Sigma) \). Therefore, we can again make use of the reduction in the proof of Theorem 3.7, since the reduction uses negative theories.

**Theorem 5.4.** The problem of deciding, given a Horn theory \( \Sigma \), a clause \( c \) and a nonnegative integer \( \alpha \), whether \( \sigma_\alpha(\Sigma)_c \models c \) holds is coNP-complete, even if \( P(c) = \emptyset \).

**Proof.** Since the hardness is proved similarly to Theorem 3.7, we show that the problem belongs to coNP.

Note that \( \sigma_\alpha(\Sigma)_c \models c \) if and only if there exists a model \( c \) of \( \sigma_\alpha(\Sigma)_c \) such that \( c(\cdot) = 0 \). A model \( c \) of \( \sigma_\alpha(\Sigma) \) can be represented by \( c = \bigwedge_{w \in W} w \) for some \( W \subseteq char(\sigma_\alpha(\Sigma)) \). In order to have such a representation, for each \( j \in OFF(\cdot) \), there exists a model \( u^{(j)} \) in \( char(\sigma_\alpha(\Sigma)) \) such that \( u_j = 0 \) and \( u \geq v \). This implies that there exists a \( W \) with \( |W| \leq n \). Since \( char(\sigma_\alpha(\Sigma)) \subseteq \bigcup_{w \in char(\Sigma)} N_\alpha(w) \) by Lemma 5.1, each \( w \in W \) can be represented as a neighbor of some model of \( char(\Sigma) \). By this representation of \( w \), we have a representation of \( v \) with a polynomial size, and we can check in polynomial time if \( v \) is a model of \( \sigma_\alpha(\Sigma) \). This implies that the problem belongs to coNP.

However, if \( \alpha \) or \( |N(c)| \) is small, the problem becomes tractable by algorithm **DEDUCTION-ENVELOPE-EXTERIOR-FROM-HORN-THEORY** (Algorithm 3).

The algorithm is based on a necessary and sufficient condition for \( \sigma_\alpha(\Sigma)_c \models c \), which is obtained from Lemma 5.2 by replacing all \( char(\Sigma) \)’s with \( mod(\Sigma) \)’s. It is not difficult to see that such a condition holds from the proof of Lemma 5.2.

**Proposition 5.5.** The problem of deciding, given a Horn theory \( \Sigma \), a clause \( c \) and a nonnegative integer \( \alpha \), whether \( \sigma_\alpha(\Sigma)_c \models c \) can be solved in \( O \left( \left( \binom{|N(c)|}{\alpha} \right) + \binom{|N(c)| - 1}{\alpha - 1} \left| \left| \Sigma \right| \right| + |P(c)| \right) \) time. In particular, it is polynomially solvable, if \( \alpha = O(1) \) or \( |N(c)| = O(\log \left| \left| \Sigma \right| \right|) \).

**Proof.** The correctness of the algorithm follows from the discussion after its description. For the time complexity, it is known [Dowling and Gallier 1983] that the
Algorithm 3: DEDUCTION-ENVELOPE-EXTERIOR-FROM-HORN-THEORY

Input: A Horn theory $\Sigma$, a clause $c$ and a nonnegative integer $\alpha$.
Output: Yes, if $\sigma_\alpha(\Sigma)_c \models c$; Otherwise, No.

Step 1. /* Check if there exists a model $v$ of $\Sigma$ such that $|\text{OFF}(v) \cap N(c)| < \alpha$. */
For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha + 1$ do
  Check if the theory obtained from $\Sigma$ by assigning $x_i = 1$ for $i \in N$ is satisfiable.
  If so, then output No and halt.
end(for)

Step 2. /* Check if there exists a set $S = \{ v \in \text{mod}(|\Sigma) | |\text{OFF}(v) \cap N(c)| = \alpha \}$ such that $\bigcup_{v \in S} \text{OFF}(v) \supseteq P(c)$ */
Let $J := \emptyset$.
For each $N \subseteq N(c)$ with $|N| = |N(c)| - \alpha$ do
  Compute the unique minimal satisfiable model $v$ for the theory obtained from $\Sigma$ by assigning $x_i = 1$ for $i \in N$.
  If such a model $v$ exists, update $J := J \cup \{ j \in P(c) | v_j = 0 \}$.
end(for)
If $J = P(c)$, then output NO and halt.

Step 3. Output Yes and halt. □

satisfiability problem, together with computing the unique minimal model of a Horn theory, is possible in linear time. Since the number of the iterations of for-loops in Steps 2 and 3 are bounded by $(|N(c)|)$ and $(|N(c)|)$, respectively, the algorithm requires $O\left(\left(|N(c)| + |N(c)|\right) \parallel \Sigma \parallel + |P(c)|\right)$ time. □

Example 5.6. Let us consider $\Sigma = \{ \bar{x}_1 \lor x_3, \bar{x}_2 \lor x_4, \bar{x}_2 \lor \bar{x}_3 \lor \bar{x}_4 \}$ and $c = x_1 \lor \bar{x}_2 \lor \bar{x}_3$. Since $\sigma_1(\Sigma) = \{ \bar{x}_1 \lor \bar{x}_2 \lor x_3 \lor x_4 \}$, $\sigma_1(\Sigma)_c$ is a tautology. Thus we have $\sigma_1(\Sigma)_c \not\models c$. We apply Algorithm 3 to $\Sigma, c$ and $\alpha = 1$.

In Step 1, by $N(c) = \{2, 3\}$ and $|N(c)| - \alpha + 1 = 2$, we check the theory obtained from $\Sigma$ by fixing $x_2 = x_3 = 1$. Since it is an unsatisfiable theory $\{x_4, \bar{x}_4\}$, we go to Step 2. In Step 2, we consider $\{2\}$ and $\{3\}$ as $N$ by $|N(c)| - \alpha = 1$. For $N = \{2\}$, we compute the unique minimal model (0001) of the obtained theory $\{\bar{x}_1 \lor x_4, \bar{x}_4 \lor \bar{x}_3\}$, and let $J = \{1\}$. For $N = \{3\}$, we compute the unique minimal model (0000) of the obtained theory $\{\bar{x}_2 \lor x_4, \bar{x}_2 \lor \bar{x}_4\}$, and we again have $J = \{1\}$. Since $J = P(c) = \{1\}$, we output “No” and halt.

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