On the Distributivity of LTL Specifications

MARKO SAMER and HELMUT VEITH
Technische Universität Darmstadt

In this paper, we investigate LTL specifications where $\gamma[\varphi \wedge \psi]$ is equivalent to $\gamma[\varphi] \wedge \gamma[\psi]$ independent of φ and ψ. Formulas γ with this property are called distributive queries because they naturally arise in Chan's seminal approach to temporal logic query solving (Chan 2000). As recognizing distributive LTL queries is PSPACE-complete, we consider distributive fragments of LTL defined by templates as in (Buccafurri et al. 2001). Our main result is a syntactic characterization of distributive LTL queries in terms of LTL templates: We construct a context-free template grammar LTLQ^x which guarantees that all specifications obtained from LTLQ^x are distributive, and all templates not obtained from LTLQ^x have simple non-distributive instantiations.

Categories and Subject Descriptors: F.4.1 [**Mathematical Logic and Formal Languages**]: Mathematical Logic—Temporal logic; F.4.3 [**Mathematical Logic and Formal Languages**]: Formal Languages—Classes defined by grammars or automata; D.2.4 [**Software Engineering**]: Software/Program Verification—Formal methods, Model checking

General Terms: Theory, Verification

Additional Key Words and Phrases: Constraint satisfaction, Distributivity, LTL, Query solving, Strongest solution, Syntactic characterization, Template characterization, Unique solution

1. INTRODUCTION

1.1 Distributive Specifications

When a model checker proves two invariants $\mathbf{G}\,\varphi$ and $\mathbf{G}\,\psi$ of a system, it follows that indeed the stronger invariant $\mathbf{G}(\varphi \wedge \psi)$ holds true in this system (and vice versa). Since the **G**-operator behaves similar to a universal quantifier, it is natural to say that conjunction distributes over **G**. In this paper, we investigate the question which LTL formulas satisfy the distributivity property

$$\gamma[\varphi \wedge \psi] \iff \gamma[\varphi] \wedge \gamma[\psi], \qquad (*)$$

where $\gamma[\theta]$ indicates that θ is a subformula of the LTL formula $\gamma[\theta]$. Our main result is a syntactic characterization of the distributive fragment of LTL. In fact, our focus lies on distributivity over conjunction; however, our results also give rise to a characterization of the distributive fragment over disjunction (see Remark 4.33).

Authors' address: Fachbereich Informatik, TU Darmstadt, Hochschulstr. 10, 64289 Darmstadt, Germany. E-Mail: {samer,veith}@cs.tu-darmstadt.de
The results presented in this article have been developed as part of [Samer 2004]. Material from the preliminary version [Samer and Veith 2004b] is reused with kind permission of Springer Science and Business Media.

The first important observation about the distributivity property $(*)$ is that a formula can be distributive with respect to one subformula, and non-distributive with respect to another subformula. For example, it is easy to see that for $\mathbf{G}(p_1 \wedge q_1) \wedge \mathbf{F}(p_2 \wedge q_2)$ we have distributivity with respect to $p_1 \wedge q_1$, but not with respect to $p_2 \wedge q_2$. Thus, a precise notion of distributivity requires a notation which uniquely identifies a subformula. To this end, we will use the distinguished symbol "?", which we call the *placeholder*. For an LTL formula φ with subformula θ, let γ be the expression obtained from φ by replacing θ with the placeholder. This simple notational trick—introducing the placeholder as an indicator for the position of a subformula—goes back to Chan's seminal paper on temporal logic queries [Chan 2000]. Since Chan's temporal logic queries provide a natural framework to analyze distributive specifications, we will in fact use terminology from this area throughout the paper. In particular, we call γ an LTL query and write $\gamma[\theta]$ to denote the formula obtained from γ by replacing the placeholder with θ. This gives an exact meaning to our definition $(*)$. Thus, we can say that $\mathbf{G}\,? \wedge \mathbf{F}(p_2 \wedge q_2)$ is distributive, while $\mathbf{G}(p_1 \wedge q_1) \wedge \mathbf{F}\,?$ is not. In Section 1.2, we will give background on temporal logic queries and explain why distributivity arises naturally in this context.

Besides temporal logic queries, distributivity over conjunction arises naturally in many verification contexts. Most obviously, distributivity facilitates the decomposition of specifications into simpler formulas, i.e., instead of verifying $\gamma[\bigwedge_{i \in I} \varphi_i]$, the model checker can verify each $\gamma[\varphi_i]$ with $i \in I$ separately. It is clear that in practice, a shorter specification does not necessarily result in a significant performance improvement, but the simpler specifications may become amenable to other techniques such as abstraction. For example, a model checker using existential abstraction can exploit formulas with fewer atomic propositions to construct smaller abstract models [Clarke et al. 1994; Clarke et al. 2000].

Another instance of distributivity applied to abstraction has appeared in recent work about parameterized systems [Clarke et al. 2006; 2008]. This work is devoted to environment abstraction, a verification method for systems with an unbounded number of concurrent processes. In environment abstraction, one considers quantified specifications of the form $\forall x, y.\ \varphi(x, y)$ where x and y range over process identifiers. For example, the mutual exclusion property

$$\forall x, y.\ x \neq y \rightarrow (\mathbf{AG}\,\mathrm{crit}_x \rightarrow \neg\mathrm{crit}_y)$$

says that no two processes x and y are in the critical section at the same time. Here, we can distribute one universal quantifier inside, obtaining

$$\forall x.\ (\mathbf{AG}\,\mathrm{crit}_x \rightarrow (\forall y.\ y \neq x \rightarrow \neg\mathrm{crit}_y)).$$

As shown by Clarke et al. [Clarke et al. 2006; 2008], formulas with this property give rise to very simple abstract models—models induced by a single "reference process"—and thus directly affect the complexity of model checking. Since the universal quantifiers in the parameterized specifications range over a finite number of processes, the distributivity criterion of the current paper directly applies to this setting.

On a more general note, a syntactic criterion for distributivity can also help to enhance axiomatic systems for temporal logics by extending the known class of simple equivalences between temporal logic formulas [Emerson 1990].

1.2 Distributivity and Temporal Logic Queries

The traditional picture of the model checking tool chain assumes that the specification φ is known when we launch the model checker. In practice however, model checkers are also used to *infer* properties about systems with unknown behavior by modifying φ and looking for the strongest variant of φ which holds true. In order to systematize this use of model checkers, Chan introduced the placeholder symbol "?" mentioned above [Chan 2000]. Chan defined a *temporal logic query* γ as a formula which contains one occurrence of the placeholder.[1] A solution of γ in a structure $\mathfrak{K}$ is a formula φ satisfying $\mathfrak{K} \models \gamma[\varphi]$. Given a structure $\mathfrak{K}$ and a query γ, the task of a query solver is to find solutions of γ in $\mathfrak{K}$.

Concentrating on CTL queries, Chan presented a symbolic algorithm which exploits the fact that certain queries always have a unique strongest solution and proposed a syntactic class of CTL queries with this property. In this paper, we return to Chan's original intention of designing a query language with unique strongest solutions. We will refer to such queries as "*exact*" queries. The fragment given by Chan is easily seen not to cover all exact CTL queries, and in fact contains errors [Samer and Veith 2003].

The results of the current paper settle Chan's question for the case of LTL queries, exploiting the close relationship between exact and distributive queries. It was noticed already by Chan that distributivity enables us to combine two solutions φ and ψ into a new solution $\varphi \wedge \psi$. In fact, by Theorem 3.19 below, *distributivity characterizes the exact queries.* Consequently, a characterization of distributive LTL specifications is tantamount to a characterization of exact LTL queries.

Proviso. *In the rest of this paper, we will mainly use the terminology of query solving, as this provides for more natural proofs and notations.*

1.3 Syntactic Characterizations of Exact/Distributive Queries

Finding a syntactic criterion for exactness/distributivity is hard because deciding this property is ExpTime-complete for CTL queries and PSpace-complete for LTL queries. Consequently, there can be no context-free or other reasonably simple grammar which recognizes all exact queries. Notwithstanding this principal limitation, we provide a syntactic characterization of exact LTL queries (or, equivalently, distributive LTL specifications) in the following sense:

We consider *query templates*, i.e., query classes where a wildcard "$\star$" is used to describe ordinary LTL subformulas which do not contain the placeholder. For example, the template $(\mathbf{X}\,?)\,\mathbf{U} \star$ stands for all queries $(\mathbf{X}\,?)\,\mathbf{U}\,\varphi$, where φ is an arbitrary LTL formula. The deterministic grammar in Table I defines the two template languages LTLQ^x and $\overline{\mathrm{LTLQ}^x}$, where the exact query templates in LTLQ^x are derived from the non-terminal $\langle EX \rangle$ and the non-exact query templates in $\overline{\mathrm{LTLQ}^x}$ are derived from the non-terminal $\langle NX \rangle$. When the context is clear we will say that a query is contained in a template language (or write $\gamma \in \mathcal{L}$) if γ is an instantiation of a query template in $\mathcal{L}$. A query is positive, if the placeholder is not in the scope

[1] Thus, the temporal logic queries considered in this paper coincide with the notation of the previous section. In the literature, other forms of queries, e.g., queries with multiple placeholders, have been considered as well.

$\langle Q1\rangle$	$::=$	$?$	$\mid$	$\star \wedge \langle Q2\rangle$	$\mid$	$\langle Q2\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q2\rangle$	$\mid$
		$\star\ \bar{\mathbf{U}}\ \langle Q2\rangle$	$\mid$	$\langle Q2\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q2\rangle$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q2\rangle$	$\mid$
		$\star \wedge \langle Q1\rangle$	$\mid$	$\star \vee \langle Q1\rangle$	$\mid$	$\mathbf{X}\ \langle Q1\rangle$	$\mid$	$\langle Q1\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$
		$\star\ \bar{\mathbf{U}}\ \langle Q1\rangle$	$\mid$	$\langle Q1\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q1\rangle$	;		
$\langle Q2\rangle$	$::=$	$\langle Q1\rangle\ \mathbf{U}\ \star$	$\mid$	$\langle Q1\rangle\ \mathbf{W}\ \star$	$\mid$	$\star \vee \langle Q2\rangle$	$\mid$	$\mathbf{X}\ \langle Q2\rangle$	$\mid$
		$\langle Q2\rangle\ \mathbf{U}\ \star$	$\mid$	$\star\ \mathbf{U}\ \langle Q2\rangle$	$\mid$	$\langle Q2\rangle\ \mathbf{W}\ \star$	$\mid$	$\star\ \mathbf{W}\ \langle Q2\rangle$	;
$\langle Q3\rangle$	$::=$	$\mathbf{F}\ \langle Q1\rangle$	$\mid$	$\mathbf{F}\ \langle Q5\rangle$	$\mid$	$\mathbf{F}\ \langle Q6\rangle$	$\mid$	$\mathbf{G}\ \langle Q4\rangle$	$\mid$
		$\mathbf{G}\ \langle Q6\rangle$	$\mid$	$\langle Q4\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$	$\star\ \mathbf{U}\ \langle Q1\rangle$	$\mid$	$\star\ \mathbf{U}\ \langle Q5\rangle$	$\mid$
		$\star\ \mathbf{U}\ \langle Q6\rangle$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q1\rangle$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q5\rangle$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q6\rangle$	$\mid$
		$\star\ \bar{\mathbf{U}}\ \langle Q4\rangle$	$\mid$	$\star\ \bar{\mathbf{U}}\ \langle Q5\rangle$	$\mid$	$\star\ \bar{\mathbf{U}}\ \langle Q6\rangle$	$\mid$	$\langle Q4\rangle\ \mathbf{W}\ \star$	$\mid$
		$\langle Q6\rangle\ \mathbf{W}\ \star$	$\mid$	$\langle Q4\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \mathbf{W}\ \langle Q5\rangle$	$\mid$	$\star\ \mathbf{W}\ \langle Q6\rangle$	$\mid$
		$\star\ \bar{\mathbf{W}}\ \langle Q4\rangle$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q5\rangle$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q6\rangle$	$\mid$	$\star \wedge \langle Q3\rangle$	$\mid$
		$\star \vee \langle Q3\rangle$	$\mid$	$\mathbf{X}\ \langle Q3\rangle$	$\mid$	$\mathbf{F}\ \langle Q3\rangle$	$\mid$	$\mathbf{G}\ \langle Q3\rangle$	$\mid$
		$\langle Q3\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$	$\star\ \mathbf{U}\ \langle Q3\rangle$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q3\rangle$	$\mid$	$\star\ \bar{\mathbf{U}}\ \langle Q3\rangle$	$\mid$
		$\langle Q3\rangle\ \mathbf{W}\ \star$	$\mid$	$\langle Q3\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \mathbf{W}\ \langle Q3\rangle$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q3\rangle$	;
$\langle Q4\rangle$	$::=$	$\langle Q3\rangle\ \mathbf{U}\ \star$	$\mid$	$\langle Q5\rangle\ \mathbf{U}\ \star$	$\mid$	$\langle Q6\rangle\ \mathbf{U}\ \star$	$\mid$	$\langle Q5\rangle\ \mathbf{W}\ \star$	$\mid$
		$\star \vee \langle Q4\rangle$	$\mid$	$\mathbf{X}\ \langle Q4\rangle$	$\mid$	$\langle Q4\rangle\ \mathbf{U}\ \star$	$\mid$	$\star\ \mathbf{U}\ \langle Q4\rangle$	$\mid$
		$\star\ \mathbf{W}\ \langle Q4\rangle$	;						
$\langle Q5\rangle$	$::=$	$\star\ \mathring{\mathbf{U}}\ \langle Q4\rangle$	$\mid$	$\star\ \mathbf{W}\ \langle Q1\rangle$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q1\rangle$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q3\rangle$	$\mid$
		$\star\ \mathring{\mathbf{W}}\ \langle Q4\rangle$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q6\rangle$	$\mid$	$\star \wedge \langle Q5\rangle$	$\mid$	$\mathbf{X}\ \langle Q5\rangle$	$\mid$
		$\langle Q5\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$	$\langle Q5\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q5\rangle$	;		
$\langle Q6\rangle$	$::=$	$\star \wedge \langle Q4\rangle$	$\mid$	$\star \vee \langle Q5\rangle$	$\mid$	$\star \wedge \langle Q6\rangle$	$\mid$	$\star \vee \langle Q6\rangle$	$\mid$
		$\mathbf{X}\ \langle Q6\rangle$	$\mid$	$\langle Q6\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$	$\langle Q6\rangle\ \mathring{\mathbf{W}}\ \star$	;		
$\langle Q7\rangle$	$::=$	$\mathbf{F}\ \langle Q2\rangle$	$\mid$	$\mathbf{F}\ \langle Q4\rangle$	$\mid$	$\mathbf{G}\ \langle Q1\rangle$	$\mid$	$\mathbf{G}\ \langle Q2\rangle$	$\mid$
		$\mathbf{G}\ \langle Q5\rangle$	$\mid$	$\star \wedge \langle Q7\rangle$	$\mid$	$\star \vee \langle Q7\rangle$	$\mid$	$\mathbf{X}\ \langle Q7\rangle$	$\mid$
		$\mathbf{F}\ \langle Q7\rangle$	$\mid$	$\mathbf{G}\ \langle Q7\rangle$	$\mid$	$\langle Q7\rangle\ \mathbf{U}\ \star$	$\mid$	$\langle Q7\rangle\ \mathring{\mathbf{U}}\ \star$	$\mid$
		$\star\ \mathbf{U}\ \langle Q7\rangle$	$\mid$	$\star\ \mathring{\mathbf{U}}\ \langle Q7\rangle$	$\mid$	$\star\ \bar{\mathbf{U}}\ \langle Q7\rangle$	$\mid$	$\langle Q7\rangle\ \mathbf{W}\ \star$	$\mid$
		$\langle Q7\rangle\ \mathring{\mathbf{W}}\ \star$	$\mid$	$\star\ \mathbf{W}\ \langle Q7\rangle$	$\mid$	$\star\ \mathring{\mathbf{W}}\ \langle Q7\rangle$	$\mid$	$\star\ \bar{\mathbf{W}}\ \langle Q7\rangle$	;
$\langle EX\rangle$	$::=$	$\langle Q1\rangle$	$\mid$	$\langle Q2\rangle$	$\mid$	$\langle Q7\rangle$	;		
$\langle NX\rangle$	$::=$	$\langle Q3\rangle$	$\mid$	$\langle Q4\rangle$	$\mid$	$\langle Q5\rangle$	$\mid$	$\langle Q6\rangle$	;

Table I. LTLQ^m production rules. The non-terminal $\langle EX\rangle$ defines the exact query templates, and the non-terminal $\langle NX\rangle$ defines the non-exact query templates.

of negation, i.e., we assume negation normal form by default. Our main result is then stated as follows:

Classification Theorem.

(*1*) *All queries derived from templates in* $LTLQ^x$ *are exact/distributive.*

(*2*) *Each template in* $\overline{LTLQ^x}$ *has a non-exact instance. Moreover, the non-exact instances are obtained by instantiating the wildcards by atomic propositions.*

(*3*) *Together,* $LTLQ^x$ *and* $\overline{LTLQ^x}$ *constitute all positive single-placeholder LTL queries over the set of temporal operators* $\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U}, \mathring{\mathbf{U}}, \bar{\mathbf{U}}, \mathbf{W}, \mathring{\mathbf{W}}, \bar{\mathbf{W}}$.

Several comments are in place to explain the range and restrictions of this result:

—It is well known that the fragment of LTL based on $\mathbf{X}$ and $\mathbf{U}$ already achieves the expressive power of LTL. This is not the case for the LTL query language because of the restriction to a single placeholder. For example, adding the operator $\varphi \, \mathring{\mathbf{U}} \, \psi = \varphi \, \mathbf{U} \, (\varphi \wedge \psi)$ makes it possible to formulate a query $? \, \mathring{\mathbf{U}} \, \psi$. The direct formulation $? \, \mathbf{U} \, (? \wedge \psi)$ however has two occurrences of the placeholder, and is therefore not in the language. The concrete choice of temporal operators follows Chan's paper.

—A similar comment applies to distributivity of specifications. This notion, too, depends on the temporal operators in the LTL version considered.

—Our characterization also holds true for arbitrary subsets of the considered operators by removing the corresponding grammar elements.

—Since negation in front of the placeholder can be removed without loss of generality, we can assume that all queries are positive; consequently, part (3) of the Classification Theorem is no restriction of generality for all fragments of our query language allowing a negation normal form. Note however, that if we allowed multiple occurrences of the same placeholder, then we would have to distinguish between positive and negative occurrences.

—The dependency graph in Figure 1 shows the relationships between the nonterminals of the grammar in Table I. Interestingly, an exact query can be extended to a non-exact query, and vice versa. In combination with the cyclicity of the dependency graph, this situation renders the inductive proofs required for parts (1) and (2) of the theorem non-trivial.

Templates are a natural concept for the syntactic characterization of logic formulas. For example, Buccafurri et al. [Buccafurri et al. 2001] have given a template characterization for the fragment of ACTL with linear counterexamples; more generally, all prefix characterizations which are very common in formal logic amount to a form of template characterization. Note, however, that a priori we cannot exclude the existence of a simple (e.g., context-free) syntactic class of LTL queries which contains all exact queries *up to logical equivalence* similar to Maidl's characterization of ACTL $\cap$ LTL [Maidl 2000]. Since deciding logical equivalence of LTL formulas is PSPACE-complete, such a characterization would not contradict the PSPACE-completeness of deciding exactness of LTL queries. While such a characterization would be semantically stronger than ours, we believe that it will be hard to find.

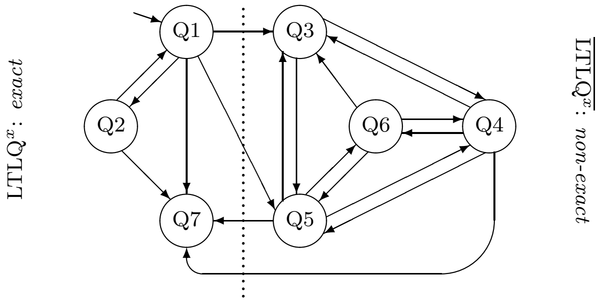

Fig. 1. LTLQ^m dependence diagram

1.4 Related Work

There has been active research in recent years on temporal logic queries. As mentioned above, the starting point was the work of Chan [Chan 2000] who focused on a syntactic fragment of exact CTL queries for which he presented an efficient symbolic query solving algorithm. Bruns and Godefroid [Bruns and Godefroid 2001] generalized Chan's idea by an automata-theoretic approach that allows to solve queries of any temporal logic having a translation to alternating automata. Gurfinkel et al. [Gurfinkel et al. 2002; Gurfinkel et al. 2003; Chechik and Gurfinkel 2003], on the other hand, investigated CTL query solving using multi-valued model checking. Hornus and Schnoebelen [Hornus and Schnoebelen 2002] studied the complexity of query solving for fragments of CTL*. Note that, in general, temporal logic queries may have multiple incomparable solutions; algorithms for computing multiple solutions as well as extensions to multiple placeholders have been the subject of [Bruns and Godefroid 2001; Hornus and Schnoebelen 2002; Gurfinkel et al. 2002; Gurfinkel et al. 2003; Chechik and Gurfinkel 2003].

In previous work on CTL [Samer and Veith 2003; 2005], we have corrected an error in Chan's original definition and extended his class to a richer fragment, but no syntactic characterization of exact CTL queries has been achieved so far despite significant efforts. Further results on temporal logic queries including relations to vacuity detection and the extended abstract of the current article can be found in [Samer 2004; Samer and Veith 2004a; 2004b; 2007].

1.5 Overview

The remainder of this paper is organized as follows: In Section 2, we start with the preliminaries. Afterwards, in Section 3, we state fundamental properties of temporal logic query languages; in particular, we show that a query is exact if and only if it is distributive and we prove that deciding whether an LTL query is exact is PSPACE-complete. Our main results follow then in Section 4. The most challenging part of the exactness proof in Section 4.1 and the counterexample construction in Section 4.2 is finding the intermediate induction hypotheses, which are necessary to make the proofs go through. Together, the results of Section 4.1 and Section 4.2 yield our syntactic characterization of exact/distributive LTL queries. Finally, Section 5 concludes the paper.

2. PRELIMINARIES

We assume the reader is familiar with the basics of model checking, i.e., Kripke structures and the linear temporal logic LTL based on the temporal operators $\mathbf{X}$ ("next"), $\mathbf{F}$ ("future"), $\mathbf{G}$ ("global"), and $\mathbf{U}$ ("until").

Let $\mathfrak{K} = (\mathcal{Q}, \mathcal{Q}_0, \delta, \ell)$ be a *Kripke structure* over the set $\mathcal{A}$ of atomic propositions, where $\mathcal{Q}$ is a set of states, $\mathcal{Q}_0 \subseteq \mathcal{Q}$ is the set of initial states, $\delta : \mathcal{Q} \to 2^{\mathcal{Q}}$ is a total transition function, and $\ell : \mathcal{Q} \to 2^{\mathcal{A}}$ is a total labeling function. A *computation path* π in $\mathfrak{K}$ is an infinite sequence of states $\pi : \mathbb{N} \to \mathcal{Q}$ such that $\pi(0) \in \mathcal{Q}_0$ and for all $s, s' \in \mathcal{Q}$ and $i \in \mathbb{N}$ it holds that $s' \in \delta(s)$ if $\pi(i) = s$ and $\pi(i+1) = s'$. An infinite sequence of states $\sigma : \mathbb{N} \to \mathcal{Q}$ is called a *computation path suffix* or simply a *path* if there exists of a computation path π and an $n \in \mathbb{N}$ such that $\pi(n+i) = \sigma(i)$ for all $i \in \mathbb{N}$. We write π^n to denote the computation path suffix of the computation path π if $\pi(n+i) = \pi^n(i)$ for all $i \in \mathbb{N}$. As usual, we write $\mathfrak{K}, \pi \models \varphi$ to denote that the LTL formula φ is satisfied on path π in $\mathfrak{K}$, and we write $\mathfrak{K} \models \varphi$ to denote $\mathfrak{K}, \pi \models \varphi$ for all computation paths π in $\mathfrak{K}$. For simplicity, we will omit $\mathfrak{K}$ if it is clear from the context, i.e., we will write $\pi \models \varphi$ instead of $\mathfrak{K}, \pi \models \varphi$.

2.1 Additional Temporal Operators

Recall from the introduction that our syntactic characterization is restricted to temporal logic queries where only a single occurrence of the placeholder is allowed. In order not to lose all queries with multiple occurrences of the placeholder, we use some additional temporal operators following Chan [Chan 2000]. In particular, we use the weak until operator $\varphi \, \mathbf{W} \, \psi \;\Leftrightarrow\; (\mathbf{G}\, \varphi) \vee (\varphi \, \mathbf{U} \, \psi)$. The other operators are variants of the strong until operator $\mathbf{U}$ and the weak until operator $\mathbf{W}$:

- The *overlapping strong until* operator: $\varphi \, \mathring{\mathbf{U}} \, \psi \;\equiv\; \varphi \, \mathbf{U} \, (\varphi \wedge \psi)$
- The *disjoint strong until* operator: $\varphi \, \bar{\mathbf{U}} \, \psi \;\equiv\; \varphi \, \mathbf{U} \, (\neg\varphi \wedge \psi)$
- The *overlapping weak until* operator: $\varphi \, \mathring{\mathbf{W}} \, \psi \;\equiv\; \varphi \, \mathbf{W} \, (\varphi \wedge \psi)$
- The *disjoint weak until* operator: $\varphi \, \bar{\mathbf{W}} \, \psi \;\equiv\; \varphi \, \mathbf{W} \, (\neg\varphi \wedge \psi)$

Of course, these temporal operators do not increase the expressive power of LTL, however, they enable us to express a certain class of temporal logic queries with multiple occurrences of the placeholder. For example, the LTL query $?\, \mathbf{U} \, (? \wedge \psi)$ would not be expressible by using the standard temporal operators when only a single occurrence of the placeholder is allowed. By using the overlapping strong until operator, however, we are able to express the query $?\, \mathring{\mathbf{U}} \, \psi \;\equiv\; ?\, \mathbf{U} \, (? \wedge \psi)$.

It is easy to verify that the following equivalences hold, which we will use in the proofs later on:

- $\varphi \, \mathring{\mathbf{W}} \, \psi \;\Leftrightarrow\; (\mathbf{G}\, \varphi) \vee (\varphi \, \mathring{\mathbf{U}} \, \psi)$
- $\varphi \, \bar{\mathbf{W}} \, \psi \;\Leftrightarrow\; (\mathbf{G}\, \varphi) \vee (\varphi \, \bar{\mathbf{U}} \, \psi)$
- $\varphi \, \mathbf{W} \, \psi \;\Leftrightarrow\; (\varphi \vee \psi) \, \mathring{\mathbf{W}} \, \psi$
- $\varphi \, \mathring{\mathbf{U}} \, \psi \;\Leftrightarrow\; (\mathbf{F}\, \psi) \wedge (\varphi \, \mathring{\mathbf{W}} \, \psi)$
- $\varphi \, \bar{\mathbf{U}} \, \psi \;\Leftrightarrow\; (\mathbf{F}\, \psi) \wedge (\varphi \, \bar{\mathbf{W}} \, \psi)$
- $\varphi \, \mathbf{U} \, \psi \;\Leftrightarrow\; (\varphi \vee \psi) \, \mathring{\mathbf{U}} \, \psi$

Note that $\varphi \, \mathring{\mathbf{W}} \, \psi \;\equiv\; \varphi \, \mathbf{W} \, (\varphi \wedge \psi) \;\Leftrightarrow\; \psi \, \mathbf{R} \, \varphi$, i.e., the overlapping weak until operator is the same as the release operator with operands swapped. Hence, when using the additional temporal operators above, the release operator can be omitted.

2.2 Negation

In addition to a restriction to a single occurrence of the placeholder, we consider only queries in *negation normal form (NNF)*, i.e., queries where negation appears only in front of atomic propositions and the placeholder. However, in order to be able to transform any temporal logic query into NNF and therefore to preserve expressive power, the chosen query language must be closed under negation, i.e., the negation of each operator in the language can be expressed by other operators in the language. The following list shows the equivalences between temporal operators and their dual operators concerning negation:

- $\neg\mathbf{F}\,\varphi \Leftrightarrow \mathbf{G}\,\neg\varphi$
- $\neg\mathbf{G}\,\varphi \Leftrightarrow \mathbf{F}\,\neg\varphi$
- $\neg(\varphi\,\mathbf{U}\,\psi) \Leftrightarrow \neg\psi\,\mathring{\mathbf{W}}\,\neg\varphi$
- $\neg(\varphi\,\mathring{\mathbf{W}}\,\psi) \Leftrightarrow \neg\psi\,\mathbf{U}\,\neg\varphi$
- $\neg(\varphi\,\mathbf{W}\,\psi) \Leftrightarrow \neg\psi\,\mathring{\mathbf{U}}\,\neg\varphi$
- $\neg(\varphi\,\mathring{\mathbf{U}}\,\psi) \Leftrightarrow \neg\psi\,\mathbf{W}\,\neg\varphi$

These equivalences follow directly from the definitions of the operators and can be easily verified. An additional equivalence not shown above is that of the next operator, which is dual to itself, that is, $\neg\mathbf{X}\,\varphi \Leftrightarrow \mathbf{X}\,\neg\varphi$.

Special cases are the disjoint strong and weak until operators, which do not have a dual operator in the above sense in our language. However, their negations can be expressed by:

- $\neg(\varphi\,\bar{\mathbf{U}}\,\psi) \Leftrightarrow (\varphi \vee \neg\psi)\,\mathring{\mathbf{W}}\,\neg\varphi$
- $\neg(\varphi\,\bar{\mathbf{W}}\,\psi) \Leftrightarrow (\varphi \vee \neg\psi)\,\mathring{\mathbf{U}}\,\neg\varphi$

Note that the first argument of both operators is duplicated after negation. In particular, this means that since we allow only a single occurrence of the placeholder it must not occur in the first argument of these operators if they are under the scope of negation. Otherwise, there would be several occurrences of the placeholder after building the NNF, e.g., $\neg(?\,\bar{\mathbf{U}}\,\psi) \Leftrightarrow (? \vee \neg\psi)\,\mathring{\mathbf{W}}\,\neg ?$. Of course, we could overcome this restriction by introducing new operators; however, placeholders in the first arguments of the disjoint strong and weak until operators are also forbidden for monotonicity reasons as will be shown below. Moreover, note that negation does not appear in the grammar defined in Table I. This, however, is no restriction since after transforming a query into NNF according to the rules above, the negation in front of the placeholder can be neglected without loss of generality.

2.3 Monotonicity

Let us consider a further property of these operators, namely monotonicity, that is, $\varphi \Rightarrow \psi$ implies $\gamma[\varphi] \Rightarrow \gamma[\psi]$ for all φ and ψ. The easiest way to obtain monotonic queries is to construct them by composing queries that are already known to be monotonic (cf. Lemma 3.6). Thus, since the simplest queries are those obtained from the temporal operators by replacing one of their argument with the placeholder, it is necessary to know which of them are monotonic. This, however, can be easily proven. For example, to show that the strong until operator is monotonic in its second argument, consider the query $\gamma = \theta\,\mathbf{U}\,?$ and assume that $\varphi \Rightarrow \psi$ as well as $\pi \models \gamma[\varphi]$ for any formulas φ and ψ and any path π. By the definition of the until operator, this means that there exists $n \in \mathbb{N}$ such that $\pi^n \models \varphi$ and for all $i < n$ it holds that $\pi^i \models \theta$. Thus, since $\varphi \Rightarrow \psi$, we know that $\pi^n \models \psi$ and therefore $\pi \models \gamma[\psi]$. Hence, the strong until operator is monotonic in its second argument.

In the same way it can be easily shown that most temporal operators introduced above are monotonic in all their arguments. The only exceptions are the disjoint strong and the disjoint weak until operator, which are not monotonic in their first argument as shown in the following example.

***Example* 2.1.** Let $\gamma_1 = ?\,\bar{\mathbf{U}}\,c$ and $\gamma_2 = ?\,\bar{\mathbf{W}}\,c$. Moreover, let π be a path such that $\ell(\pi(0)) = \{a, c\}$ and $\ell(\pi(i)) = \emptyset$ for all $i \geq 1$. Then, it can be easily verified that $\pi \models \gamma_1[a \wedge b]$ as well as $\pi \models \gamma_2[a \wedge b]$, but $\pi \not\models \gamma_1[a]$ and $\pi \not\models \gamma_2[a]$ although $a \wedge b \Rightarrow a$. Note that this is a counterexample to monotonicity in the sense of *monotonically increasing*. In order to show that both queries are not monotonically decreasing either, let π be a path such that $\ell(\pi(0)) = \{a\}$ and $\ell(\pi(i)) = \{c\}$ for all $i \geq 1$. Then, it can be easily verified that $\pi \models \gamma_1[a \vee b]$ as well as $\pi \models \gamma_2[a \vee b]$, but $\pi \not\models \gamma_1[b]$ and $\pi \not\models \gamma_2[b]$ although $b \Rightarrow a \vee b$. Hence, γ_1 and γ_2 are neither monotonically increasing nor monotonically decreasing.

3. PROPERTIES OF TEMPORAL LOGIC QUERIES

In this section, we study fundamental properties of temporal logic queries and provide alternative characterizations of exactness. In particular, we show the equivalence between exactness and distributivity and we prove that deciding exactness of LTL queries is PSPACE-complete. Note that most of the results in this section are not restricted to LTL.

In the following definition we allow temporal logic queries to contain multiple occurrences of the placeholder. This yields more general results in the current section. In Section 4 we will restrict our considerations again to temporal logic queries with a single occurrence of the placeholder as described in the introduction.

***Definition* 3.1** (*Temporal logic query*)**.** A *temporal logic query* is a temporal logic formula where some subformulas are replaced by a special variable "?", called *placeholder*. We denote the set of LTL queries by LTLQ.

It is straightforward to generalize this definition in such a way that a query can contain different variables. For simplicity, however, we restrict our considerations to queries with a single variable, namely the placeholder. In the following, we will simply write *query* instead of *temporal logic query* and *structure* instead of *Kripke structure*.

***Definition* 3.2** (*Solution*)**.** Let γ be a query, $\mathfrak{K}$ be a structure, and φ be a formula. We write $\gamma[\varphi]$ to denote the result of substituting all occurrences of the placeholder in γ by φ. If $\mathfrak{K} \models \gamma[\varphi]$, then we say that φ is a *solution* of γ in $\mathfrak{K}$. We denote the set of all solutions of γ in $\mathfrak{K}$ by $\mathrm{sol}(\mathfrak{K}, \gamma) = \{\varphi \mid \mathfrak{K} \models \gamma[\varphi]\}$.

***Example* 3.3.** Consider the structure $\mathfrak{K}$ shown in Figure 2 and let $\gamma_1 = a\,\mathbf{U}\,\mathbf{X}(c \vee \mathbf{G}\,?)$ and $\gamma_2 = ?\,\mathbf{U}\,\mathbf{G}\,?$ be LTL queries. It can be easily checked that $\mathfrak{K} \models \gamma_1[b \wedge \mathbf{X}\,d]$ and $\mathfrak{K} \models \gamma_2[a \vee \mathbf{X}\,d]$. Hence, $b \wedge \mathbf{X}\,d$ and $a \vee \mathbf{X}\,d$ are solutions of γ_1 and γ_2, respectively, in $\mathfrak{K}$.

A query γ can be seen as a function $\gamma : \varphi \mapsto \gamma[\varphi]$ that maps formulas to formulas. This interpretation leads to an important and natural property of queries, namely monotonicity. The following definition of monotonic queries originates from

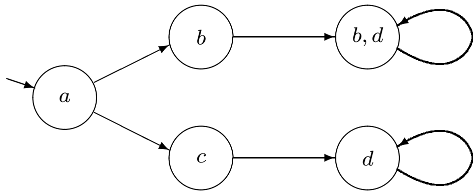

Fig. 2. Query solving example

Chan [Chan 2000]. Note that *monotonic* means *monotonically increasing*; the case of *monotonically decreasing* queries is symmetric.

***Definition* 3.4** (*Monotonic query).* A query γ is *monotonic* if $\varphi \Rightarrow \psi$ implies $\gamma[\varphi] \Rightarrow \gamma[\psi]$ for all formulas φ and ψ.

Two basic properties of monotonic queries are now stated in the following lemma.

Lemma 3.5 ([Chan 2000])**.** *Let γ be a monotonic query and $\mathfrak{K}$ be a structure. Then*

(*1*) *The query γ has a solution in $\mathfrak{K}$ iff $\mathfrak{K} \models \gamma[\top]$.*

(*2*) *Every formula is a solution of γ in $\mathfrak{K}$ iff $\mathfrak{K} \models \gamma[\bot]$.*

Proof. The *if* direction of *1* and the *only if* direction of *2* are trivial. For the *only if* direction of *1* suppose that $\mathfrak{K} \models \gamma[\varphi]$ for some formula φ. Since $\varphi \Rightarrow \top$, we obtain by monotonicity $\mathfrak{K} \models \gamma[\top]$. For the *if* direction of *2* suppose that $\mathfrak{K} \models \gamma[\bot]$. Since $\bot \Rightarrow \varphi$ for every formula φ, we obtain by monotonicity $\mathfrak{K} \models \gamma[\varphi]$ for every formula φ. □

Another important and well-known property is that the composition of monotonic queries is also monotonic.

Lemma 3.6. *Let γ and γ' be monotonic queries. Then, $\gamma[\gamma']$ is monotonic.*

Proof. Let φ and ψ be formulas such that $\varphi \Rightarrow \psi$. Since γ' is monotonic, we know that $\gamma'[\varphi] \Rightarrow \gamma'[\psi]$. Thus, since γ is monotonic, we know that $\gamma[\gamma'[\varphi]] \Rightarrow \gamma[\gamma'[\psi]]$. Hence, since $\gamma[\gamma'[\theta]]$ is syntactically the same as $\gamma[\gamma'][\theta]$ for all formulas θ, we obtain $\gamma[\gamma'][\varphi] \Rightarrow \gamma[\gamma'][\psi]$, i.e., $\gamma[\gamma']$ is monotonic. □

To obtain the maximum information a query provides, it is necessary to consider *all* its solutions. However, since the number of solutions can be very large in general, it is desirable to have strongest solutions that subsume all other solutions. We call such strongest solutions *minimal solutions.* However, in the following we are interested in a more specific solution; in fact, we are interested in the case where exactly one minimal solution exists. We call this minimal solution the *least solution*.[2]

***Definition* 3.7** (*Least solution).* A solution μ of a query γ in a structure $\mathfrak{K}$ is the *least solution* if for every solution φ of γ in $\mathfrak{K}$ it holds that $\mu \Rightarrow \varphi$.

In the following, we define queries that always have a least solution.

[2]Note that the set of solutions of a query together with logical implication form a partial ordered set. The minimal solutions are the minimal elements of this set. The least solution (if it exists) is the least element of this set.

***Definition* 3.8** (*Bounded query*)**.** A query is *bounded* if it has a least solution in every structure where the set of solutions is not empty.

***Remark* 3.9.** Note that not every monotonic query is bounded. For example, let $\gamma = \mathbf{F}\,?$ be an LTL query and π be a path such that $\ell(\pi(0)) = \{p\}$ and $\ell(\pi(i)) = \{q\}$ for all $i \geq 1$. It is easy to see that γ is monotonic and that p and q are solutions of γ on π, that is, $\pi \models \gamma[p] \wedge \gamma[q]$. Now suppose that there exists a least solution μ of γ on π. Then, we know that $\mu \Rightarrow p$ and $\mu \Rightarrow q$, which is trivially equivalent to $\mu \Rightarrow p \wedge q$. Thus, by monotonicity, it follows that $p \wedge q$ must also be a solution of γ on π. However, it is easy to see that $\pi \not\models \gamma[p \wedge q]$. Hence, γ is monotonic but not bounded.

One the other hand, note also that not every bounded query is monotonic. For example, let $\gamma = \,?\,\bar{\mathbf{U}}\,c$ be an LTL query and π be a path such that $\ell(\pi(i)) = \{c, p\}$ for all $i \in \mathbb{N}$. In order to see that γ is bounded, note that the conjunction of all solutions is always a least solution. So it is sufficient to show that $\gamma[\varphi] \wedge \gamma[\psi] \Rightarrow \gamma[\varphi \wedge \psi]$ for all formulas φ and ψ. Suppose that $\sigma \models \gamma[\varphi] \wedge \gamma[\psi]$ for any path σ. By definition, this implies that there exist unique numbers k and l such that $\sigma^{[0,k)} \models \varphi$, $\sigma^{[0,l)} \models \psi$, $\sigma^k \models \neg\varphi \wedge c$, and $\sigma^l \models \neg\psi \wedge c$. Thus, it follows that $\sigma^{[0,\min(k,l))} \models \varphi \wedge \psi$ and $\sigma^{\min(k,l)} \models \neg(\varphi \wedge \psi) \wedge c$, which implies $\sigma \models \gamma[\varphi \wedge \psi]$. So we have shown that γ is bounded. However, it is easy to see that $p \wedge q$ is a solution of γ on π, whereas p is not a solution. Hence, γ is bounded but not monotonic.

Now let us introduce a special solution that exactly characterizes all other solutions of a query. We call such a solution an *exact solution*.

***Definition* 3.10** (*Exact solution*)**.** A solution ξ of a query γ in a structure $\mathfrak{K}$ is *exact* if it holds that φ is a solution of γ in $\mathfrak{K}$ iff $\xi \Rightarrow \varphi$.

The following proposition connects monotonic queries, least solutions, and exact solutions. It is directly implied by the definitions.

Proposition 3.11. *If a monotonic query has a least solution, then it is exact.*

The following definition introduces the kind of queries in which we are primarily interested in this chapter. Their set of solutions—provided that it is not empty—can always be exactly characterized by a single formula.

***Definition* 3.12** (*Exact query*)**.** A query is *exact* if it has an exact solution in every structure where the set of solutions is not empty.

By putting the above results together, we obtain the equivalence between exact queries and queries that are both bounded and monotonic.

Theorem 3.13. *A query is exact iff it is bounded and monotonic.*

Proof. For the *if* direction, let γ be a bounded and monotonic query. Thus, since γ is bounded, we know that there always exists a least solution if the set of solutions is not empty. Hence, by Proposition 3.11, it follows that γ is exact. For the *only if* direction, let γ be an exact query. Thus, we know that there always exists an exact solution if the set of solutions is not empty. It is easy to see that every exact solution is also a least solution. Hence, γ is bounded. Now let ξ be the exact solution of γ in some structure $\mathfrak{K}$ where the set of solutions is not empty.

Suppose that $\varphi \Rightarrow \psi$ and $\mathfrak{K} \models \gamma[\varphi]$, i.e., φ is a solution of γ in $\mathfrak{K}$. Since ξ is exact, we know that $\xi \Rightarrow \varphi$ and therefore $\xi \Rightarrow \psi$. Thus, ψ must also be a solution of γ in $\mathfrak{K}$, that is, $\mathfrak{K} \models \gamma[\psi]$. Hence, γ is monotonic. □

***Remark* 3.14.** Note that the general results of Chan [Chan 2000] are based on the assumption that the considered queries are monotonic. So he did not say anything about the existence of exact queries beyond the class of monotonic queries. Our above result, however, implies that there exist no exact queries that are not monotonic.

In order to present an alternative characterization of exact queries, we need the following important properties of queries.

Definition* 3.15** (*Collecting, Separating, Distributive).** A query γ is *collecting* if it satisfies $\gamma[\varphi] \wedge \gamma[\psi] \Rightarrow \gamma[\varphi \wedge \psi]$, and *separating* if it satisfies $\gamma[\varphi \wedge \psi] \Rightarrow \gamma[\varphi] \wedge \gamma[\psi]$ for all formulas φ and ψ. A query is *distributive (over conjunction)* if it is collecting and separating.

Now we will show several relations between the above properties. Let us start with a relation between collecting and bounded queries.

Lemma 3.16. *Every collecting query is bounded.*

Proof. Let γ be a collecting query and $\mathfrak{K}$ be a structure. Consider the set $\mathcal{S}$ of all solutions of γ in $\mathfrak{K}$, that is, $\mathfrak{K} \models \bigwedge_{\varphi \in \mathcal{S}} \gamma[\varphi]$. Thus, since γ is collecting, it follows that $\mathfrak{K} \models \gamma[\bigwedge \mathcal{S}]$. Moreover, since $\bigwedge \mathcal{S} \Rightarrow \varphi$ for every $\varphi \in \mathcal{S}$, we know that $\bigwedge \mathcal{S}$ is a least solution of γ in $\mathfrak{K}$. Hence, γ is bounded. □

The below result states that monotonicity and separability are equivalent.

Lemma 3.17. *A query is separating iff it is monotonic.*

Proof. Let γ be a query. For the *if* direction, consider the valid implications $\varphi \wedge \psi \Rightarrow \varphi$ as well as $\varphi \wedge \psi \Rightarrow \psi$. Thus, by monotonicity, we obtain $\gamma[\varphi \wedge \psi] \Rightarrow \gamma[\varphi]$ as well as $\gamma[\varphi \wedge \psi] \Rightarrow \gamma[\psi]$, which is trivially equivalent to $\gamma[\varphi \wedge \psi] \Rightarrow \gamma[\varphi] \wedge \gamma[\psi]$. Hence, γ is separating. For the *only if* direction, suppose that $\varphi \Rightarrow \psi$. Since $\varphi \Rightarrow \psi$ implies $\varphi \Leftrightarrow \varphi \wedge \psi$, we know that $\gamma[\varphi]$ is equivalent to $\gamma[\varphi \wedge \psi]$. Thus, since γ is separating, we obtain $\gamma[\varphi] \Leftrightarrow \gamma[\varphi \wedge \psi] \Rightarrow \gamma[\psi]$. Hence, γ is monotonic. □

Finally, let us consider the relation between exact and collecting queries.

Lemma 3.18. *Every exact query is collecting.*

Proof. Let γ be an exact query and ξ be its exact solution in a structure $\mathfrak{K}$ where the set of solutions is not empty. Further, let φ and ψ be any solutions of γ in $\mathfrak{K}$, that is, $\mathfrak{K} \models \gamma[\varphi] \wedge \gamma[\psi]$. By definition, we know that $\xi \Rightarrow \varphi$ and $\xi \Rightarrow \psi$, which is trivially equivalent to $\xi \Rightarrow \varphi \wedge \psi$. Thus, $\varphi \wedge \psi$ must be a solution of γ in $\mathfrak{K}$, that is, $\mathfrak{K} \models \gamma[\varphi \wedge \psi]$. Hence, γ is collecting. □

By putting the above auxiliary results together, we obtain an alternative characterization of exact queries by the following theorem.

Theorem 3.19. *A query is exact iff it is distributive.*

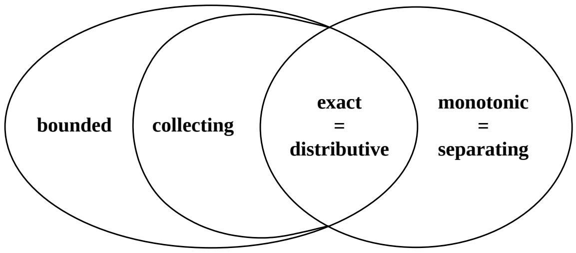

Fig. 3. Query classes

PROOF. For the *if* direction, we know by Lemma 3.16 and Lemma 3.17 that every distributive query is bounded and monotonic. Hence, by Theorem 3.13, it follows that every distributive query is exact. For the *only if* direction, we know by Theorem 3.13 and Lemma 3.17 that every exact query is separating. In addition, by Lemma 3.18, we know that every exact query is collecting. Hence, every exact query is distributive. □

The following result will be the basis of our syntactic characterization of exact LTL queries in Section 4. It is directly implied by the previous theorem and the equivalence between separability and monotonicity according to Lemma 3.17.

Corollary 3.20. *A query is exact iff it is monotonic and collecting.*

***Remark* 3.21.** Note that this equivalence provides a naive algorithm for computing exact solutions of exact queries by building the conjunction of all solutions. Due to the collecting property, we know that the resulting formula must be a solution which trivially is a least solution. Thus, by Proposition 3.11, we know that it is also an exact solution.

An overview of the query classes introduced above is given in Figure 3. Let us now consider LTL queries. The following theorem shows that deciding whether a given LTL query is exact is PSPACE-complete. In the proof we use a similar idea as in the proof of the validity of CTL queries by Chan [Chan 2000]. Note that in contrast to Chan, we use only the standard temporal operators in this reduction. Moreover, note that the placeholder occurs only once in the used queries. Hence, the complexity remains unchanged when restricting the query language to the standard temporal operators and by allowing only a single occurrence of the placeholder.

Theorem 3.22. *Deciding whether a given LTL query is exact is* PSPACE*-complete.*

PROOF. We show that deciding exactness of LTL queries is polynomial-time equivalent to deciding validity of LTL formulas. To this aim, it suffices by Theorem 3.19 to show that deciding distributivity of LTL queries is equivalent to deciding validity of LTL formulas. Then, we obtain our result by the fact that deciding validity of LTL formulas is PSPACE-complete [Sistla and Clarke 1985].

For the reduction to LTL validity, note that an LTL query γ is distributive iff the formula $\gamma[p] \wedge \gamma[q] \leftrightarrow \gamma[p \wedge q]$ is valid for some atomic propositions p and q not occurring in γ. The *if* direction follows by contraposition when assuming that γ is not distributive, i.e., there exists a structure $\mathfrak{K}$ and formulas φ and ψ such that $\mathfrak{K} \not\models \gamma[\varphi] \wedge \gamma[\psi] \leftrightarrow \gamma[\varphi \wedge \psi]$. Let π be a counterexample path in $\mathfrak{K}$ such that $\pi \not\models \gamma[\varphi] \wedge \gamma[\psi] \leftrightarrow \gamma[\varphi \wedge \psi]$. Now we construct a new structure $\mathfrak{K}'$ from π by labeling states in π with new atomic propositions p and q iff φ and ψ, respectively, hold at these states. Then, it obviously holds that $\mathfrak{K}' \not\models \gamma[p] \wedge \gamma[q] \leftrightarrow \gamma[p \wedge q]$. The *only if* direction follows trivially from the definition of distributivity.

For the reduction from LTL validity, let φ be an LTL formula. Moreover, let p and q be atomic propositions not occurring in φ, and let $\gamma = (\mathbf{G}\,\varphi) \vee (p\,\mathbf{U}\,?)$ be an LTL query. Now we will show that φ is valid iff γ is distributive.

The *if* direction follows by contraposition when assuming that φ is not valid, i.e., there exists a structure $\mathfrak{K}$ such that $\mathfrak{K} \not\models \varphi$. Since p and q are not occurring in φ, we can assume w.l.o.g. that the initial state of $\mathfrak{K}$ is labeled with p, the immediate successor states of this initial state are labeled with q, and no other states of $\mathfrak{K}$ are labeled with p or q. Moreover, we can assume w.l.o.g. that the initial state is not an immediate successor state of itself. Then, it is easy to see that $\mathfrak{K} \models \gamma[p] \wedge \gamma[q]$, but $\mathfrak{K} \not\models \gamma[p \wedge q]$. Hence, γ is not collecting and therefore not distributive.

For the *only if* direction, we know that φ is valid, which implies that also $\mathbf{G}\,\varphi$ is valid. Thus, it is easy to see that $\mathfrak{K} \models \gamma[\bot]$ for all structures $\mathfrak{K}$. Hence, by Lemma 3.5, it follows trivially that γ is distributive. ☐

Therefore, since deciding whether a given LTL query is exact is very hard, we define a syntactic fragment of exact LTL queries in the remainder of this paper. More precisely, we are able to present a syntactic characterization of exact LTL queries. Since the existence of a simple grammar that characterizes all exact LTL queries is unlikely in view of Theorem 3.22, we use *query templates* in the definition of our grammar (cf. Table I). Thus, when proving that a template satisfies some property, the property has to be proven for all instantiations of the template, and when proving that a template does not satisfy some property, the absence of the property has to be proven for a single instantiation of the template. Of course, template characterizations do not capture all exact queries, but they allow an approximation that is consistent with our complexity result.

4. THE CHARACTERIZATION OF EXACT LTL QUERIES

In this section, we present LTLQ^x, an exact query language based on LTL. As mentioned before, we restrict our considerations in the following to queries with a single occurrence of the placeholder as introduced by Chan [Chan 2000]. Recall that this restriction is slightly weakened by the additional temporal operators defined in Section 2.1. Moreover, according to Section 2.2, we assume a query to be in negation normal form and with no negation in front of the placeholder.

Let us start with a class of monotonic LTL queries. To this aim, recall our observations on the monotonicity of temporal operators in Section 2.3. It is then easy to define a class of monotonic LTL queries.

***Definition* 4.1** (*LTLQ^m*)**.** The language *LTLQ^m* is the largest set of LTL queries with a single occurrence of the placeholder that do not contain a subquery of the form $\gamma\,\bar{\mathbf{U}}\,\varphi$ or $\gamma\,\bar{\mathbf{W}}\,\varphi$, where γ is some LTL query and φ is some LTL formula.

For example, the LTL query $\mathbf{X}\,(\varphi\,\bar{\mathbf{U}}\,\mathbf{G}\,?)$ is in LTLQ^m, whereas $\mathbf{X}\,((\varphi \vee ?)\,\bar{\mathbf{U}}\,\psi)$ is not in LTLQ^m. The following lemma justifies our claim from above and follows directly from Lemma 3.6 and the monotonicity of temporal operators (see Section 2.3).

Lemma 4.2. *Every query in $LTLQ^m$ is monotonic.*

Now according to Corollary 3.20 and Lemma 4.2, our next step towards an exact query language is to restrict LTLQ^m to a class of collecting queries. To this aim, LTLQ^m must be divided into sublanguages. The corresponding deterministic grammar is shown in Table I. In the following, we write LTLQ^1 for the language derived from the non-terminal $\langle Q1 \rangle$, LTLQ^2 for the language derived from the non-terminal $\langle Q2 \rangle$, and so on. It is easy to see that $\mathrm{LTLQ}^m = \bigcup_{i=1}^{7} \mathrm{LTLQ}^i$ since every operator allowed in LTLQ^m occurs in combination with every non-terminal.

***Definition* 4.3** *($LTLQ^x$).* The language $LTLQ^x$ is defined as $\mathrm{LTLQ}^x = \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^2 \cup \mathrm{LTLQ}^7$. Its complement within LTLQ^m is given by $\overline{\mathrm{LTLQ}^x} = \mathrm{LTLQ}^m \setminus \mathrm{LTLQ}^x = \mathrm{LTLQ}^3 \cup \mathrm{LTLQ}^4 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$.

The dependencies between the non-terminals in the grammar in Table I are illustrated in Figure 1. This graph can be interpreted as an automaton that analyzes a given query starting from the placeholder up to the topmost operator. Its initial state is Q1 because the placeholder occurs in the definition of $\langle Q1 \rangle$. For example, there is a transition from state Q1 to state Q7 because in the definition of non-terminal $\langle Q7 \rangle$ there appears $\mathbf{G}\,\langle Q1 \rangle$, i.e., there is an operator that leads from non-terminal $\langle Q1 \rangle$ to non-terminal $\langle Q7 \rangle$. For simplicity, we omitted the transitions from each state to itself and the labels on each transition. Since the grammar is deterministic, each query can be uniquely assigned to a node in the graph. For example, it can be easily verified that the query $(b\,\mathbf{U}\,(a \wedge ?))\,\mathbf{U}\,c$ belongs to node Q4. The nodes on the left hand side of the dotted line represent LTLQ^x and the nodes on the right hand side represent its complement $\overline{\mathrm{LTLQ}^x}$.

In the following, we will show that all instantiations of templates in LTLQ^x are exact and that to each template in $\overline{\mathrm{LTLQ}^x}$ there exists a simple instantiation that is not exact. This will be done by a series of nested inductive proofs on the sublanguages of LTLQ^m. As we shall see soon, a major complication in the proofs arises from the fact that the non-trivial dependencies between these sublanguages are circular as depicted in Figure 1.

4.1 Proof of Exactness

This section is devoted to our first main result, namely the exactness of LTLQ^x. To this aim, we show that all queries in LTLQ^1, LTLQ^2, and LTLQ^7 are collecting. Then, the exactness of LTLQ^x follows by Corollary 3.20. However, since the collecting property introduced in Definition 3.15 is too weak, we need the following stronger variants.

***Definition* 4.4** *(Collecting properties).* Let γ be an LTL query.
We say γ is *strong collecting* if for all paths π and formulas φ and ψ:
If $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$, then $\pi^n \models \gamma[\varphi \wedge \psi]$.
We say γ is *boundary collecting* if for all paths π and formulas φ and ψ:
If $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$, then $\pi^n \models \gamma[\varphi \wedge \psi]$ or $\pi \models \gamma[\bot]$.
We say γ is *intermediate collecting* if for all paths π and formulas φ and ψ:

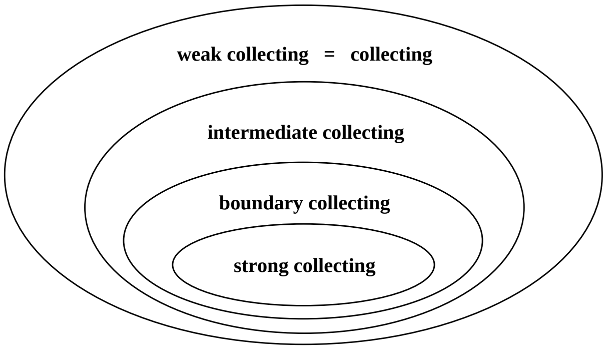

Fig. 4. Collecting queries

If $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$, then $\pi^n \models \gamma[\varphi \wedge \psi]$ or there exists $r < n$ such that $\pi^r \models \gamma[\bot]$.
We say γ is *weak collecting* if it is collecting.

Note that every strong collecting query is also boundary collecting, every boundary collecting query is also intermediate collecting (consider the case $r = 0$ together with Lemma 3.5), and every intermediate collecting query is also weak collecting (consider the case $n = 0$). This implication chain is illustrated in Figure 4.

Now let us start to prove the exactness of LTLQ^x. This will be done by a series of auxiliary results using the above collecting properties.

The following lemma is our first auxiliary result towards a proof for LTLQ^1 and LTLQ^2. Since LTLQ^1 and LTLQ^2 depend on each other (cf. Figure 1), we have to make a preliminary assumption on subqueries in LTLQ^1.

Lemma 4.5. *Let $\gamma \in$ LTLQ^2. Suppose that every subquery in LTLQ^1 is weak collecting. Then, γ is intermediate collecting.*

Proof. Structural induction on γ. See Appendix A for details. □

The following lemma is our second auxiliary result towards a proof for LTLQ^1 and LTLQ^2. This time, we have to make an assumption on subqueries in LTLQ^2.

Lemma 4.6. *Let $\gamma \in$ LTLQ^1. Suppose that every subquery in LTLQ^2 is intermediate collecting. Then, γ is weak collecting.*

Proof. Structural induction on γ. See Appendix A for details. □

In order to obtain the statement of Lemma 4.6 without its assumption, we use an inductive proof on the number of subqueries in LTLQ^2.

Lemma 4.7. *Every query in LTLQ^1 is weak collecting.*

Proof. Induction on the number of subqueries in LTLQ^2.
Induction start: If $\gamma \in$ LTLQ^1 contains no subquery in LTLQ^2, then the assumption of Lemma 4.6 is trivially satisfied. Thus, we can apply Lemma 4.6 to γ and obtain that γ is weak collecting.
Induction step: Let γ' be any LTLQ^2 subquery of γ, and γ'' be any LTLQ^1 subquery

of γ'. Note that by definition every LTLQ^2 query has an LTLQ^1 subquery. Since the number of LTLQ^2 subqueries of γ'' must be less than the number of LTLQ^2 subqueries of γ, we can apply the induction hypothesis and obtain that γ'' is weak collecting. Thus, since γ'' was chosen w.l.o.g., the assumption of Lemma 4.5 is satisfied. So we can apply Lemma 4.5 to γ' and obtain that γ' is intermediate collecting. Since γ' was chosen w.l.o.g., the assumption of Lemma 4.6 is satisfied. Hence, we can apply Lemma 4.6 to γ and obtain that γ is weak collecting. □

The following corollary is directly implied by Lemma 4.5 and Lemma 4.7.

Corollary 4.8. *Every query in LTLQ^2 is intermediate collecting.*

Now let us turn our attention to LTLQ^7. The following lemma is our first auxiliary result towards a proof for LTLQ^7. Since LTLQ^4 and LTLQ^5 depend on each other (cf. Figure 1), we have to make a preliminary assumption on subqueries in LTLQ^5.

Lemma 4.9. *Let $\gamma \in LTLQ^4$. Suppose that for every subquery $\gamma' \in LTLQ^5$ it holds that $\mathbf{G}\,\gamma'$ is weak collecting. Then, $\mathbf{F}\,\gamma$ is boundary collecting.*

Proof. Structural induction on γ. See Appendix A for details. □

The following lemma is our second auxiliary result towards a proof for LTLQ^7. This time, we have to make an assumption on subqueries in LTLQ^4.

Lemma 4.10. *Let $\gamma \in LTLQ^5$. Suppose that for every subquery $\gamma' \in LTLQ^4$ it holds that $\mathbf{F}\,\gamma'$ is weak collecting. Then, $\mathbf{G}\,\gamma$ is weak collecting.*

Proof. Structural induction on γ. See Appendix A for details. □

In order to obtain the statement of Lemma 4.9 without its assumption, we use an inductive proof on the number of subqueries in LTLQ^5.

Lemma 4.11. *Let $\gamma \in LTLQ^4$. Then, $\mathbf{F}\,\gamma$ is boundary collecting.*

Proof. Induction on the number of subqueries in LTLQ^5.
Induction start: If γ contains no subquery in LTLQ^5, then the assumption of Lemma 4.9 is trivially satisfied. Hence, we can apply Lemma 4.9 to γ and obtain that $\mathbf{F}\,\gamma$ is boundary collecting.
Induction step: Let γ' be any LTLQ^5 subquery of γ. If γ' contains no LTLQ^4 subquery, then the assumption of Lemma 4.10 is trivially satisfied. Otherwise, let γ'' be any LTLQ^4 subquery of γ'. Since the number of LTLQ^5 subqueries of γ'' must be less than the number of LTLQ^5 subqueries of γ, we can apply the induction hypothesis and obtain that $\mathbf{F}\,\gamma''$ is boundary collecting. Thus, since γ'' was chosen w.l.o.g., the assumption of Lemma 4.10 is again satisfied. So in both cases we can apply Lemma 4.10 to γ' and obtain that $\mathbf{G}\,\gamma'$ is weak collecting. Since γ' was chosen w.l.o.g., the assumption of Lemma 4.9 is satisfied. Hence, we can apply Lemma 4.9 to γ and obtain that $\mathbf{F}\,\gamma$ is boundary collecting. □

The following corollary is directly implied by Lemma 4.10 and Lemma 4.11.

Corollary 4.12. *Let $\gamma \in LTLQ^5$. Then, $\mathbf{G}\,\gamma$ is strong collecting.*

Proof. Suppose that $\pi \models \mathbf{G}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{G}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Then, it is easy to see that $\pi^n \models \mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$. Hence, since $\mathbf{G}\,\gamma$ is weak collecting by Lemma 4.10 and Lemma 4.11, we obtain $\pi^n \models \mathbf{G}\,\gamma[\varphi \wedge \psi]$. □

Note that every query in LTLQ^7 contains a subquery of the form $\mathbf{F}\,\gamma$, where $\gamma \in \mathrm{LTLQ}^2 \cup \mathrm{LTLQ}^4$, or of the form $\mathbf{G}\,\gamma$, where $\gamma \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^2 \cup \mathrm{LTLQ}^5$. Therefore, we can use Lemma 4.7, Corollary 4.8, Lemma 4.11, and Corollary 4.12 as induction start in the proof of the following lemma.

Lemma 4.13. *Every query in $LTLQ^7$ is boundary collecting.*

Proof. Structural induction on γ. See Appendix A for details. □

Now recall that LTLQ^x is defined as the union of LTLQ^1, LTLQ^2, and LTLQ^7 (see Definition 4.3). Moreover, as already mentioned, every boundary collecting query is intermediate collecting and every intermediate collecting query is (weak) collecting. Thus, we obtain the following corollary by Lemma 4.7, Corollary 4.8, and Lemma 4.13.

Corollary 4.14. *Every query in $LTLQ^x$ is collecting.*

Hence, since LTLQ^x is a subset of LTLQ^m, we obtain by Lemma 4.2, Corollary 4.14, and Corollary 3.20 the following theorem.

Theorem 4.15. *Every query in $LTLQ^x$ is exact.*

4.2 Proof of Maximality

In this section, we will show that LTLQ^x is maximal in the sense that each template in $\overline{\mathrm{LTLQ}^x}$ has a simple instantiation that is not collecting and therefore not exact. To this aim, we inductively construct a path π for each such instantiation γ such that $\pi \models \gamma[p] \wedge \gamma[q]$ but $\pi \not\models \gamma[p \wedge q]$. In particular, the instantiations are simple in the following sense.

***Definition* 4.16** (*Simple query*)**.** A query γ is *simple* if every (variable free) subformula of γ is atomic and occurs only once in γ. We denote the set of atomic propositions occurring in γ by $\mathrm{aprop}(\gamma)$.

For example, the query $\mathbf{G}\,(a\,\mathbf{U}\,(b \wedge \mathbf{X}\,?))$ is simple whereas the queries $\mathbf{G}\,((a \vee b)\,\mathbf{U}\,\mathbf{X}\,?)$ and $\mathbf{G}\,(a\,\mathbf{U}\,\mathbf{X}\,(a \wedge ?))$ are not simple because $a \vee b$ is not atomic and a occurs twice, respectively.

It is easy to see that for every subquery γ'' of a simple query $\gamma = \gamma'[\gamma'']$, it holds that $\mathrm{aprop}(\gamma') \cap \mathrm{aprop}(\gamma'') = \emptyset$. Thus, all subformulas of a simple query are independent of each other as they are different atomic propositions. This allows the inductive construction of a counterexample path by labeling the states according to the atomic propositions in a query without affecting the truth value of other subformulas.

***Example* 4.17.** Consider the simple query $\gamma = (b\,\mathbf{U}\,(a \wedge ?))\,\mathbf{U}\,c$. Since $\gamma \in \mathrm{LTLQ}^4$, there exists a path π such that $\pi \models \gamma[p] \wedge \gamma[q]$ but $\pi \not\models \gamma[p \wedge q]$ for any atomic propositions p and q not occurring in γ. The construction of π for this simple example according to our proof is illustrated in Figure 5. We start with an initial path on which for all $n \in \mathbb{N}$ it holds that $\pi^{4n} \models p$, $\pi^{4n+2} \models q$, and $\pi \models \mathbf{G}\,\neg(p \wedge q)$.

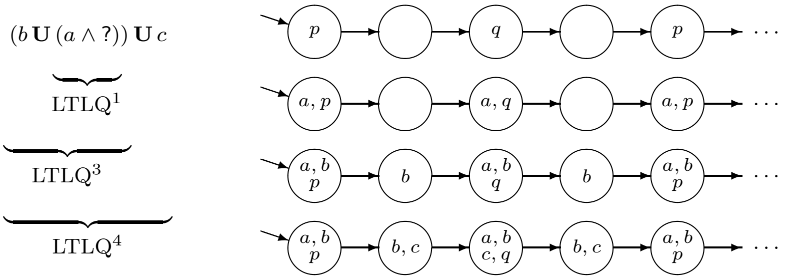

Fig. 5. Counterexample construction

Then, according to the structure of γ, we successively add labels to the states of π in such a way that we can always achieve or preserve a counterexample by adding further labels. It is easy to see that the resulting path in Figure 5 is indeed a counterexample to the collecting property. Note that there exist simpler ad hoc counterexamples to the query γ. The construction in Figure 5, however, illustrates the general method that works for all queries.

We will need the following operation for such a construction.

***Definition* 4.18** (*Concatenation*)**.** Let σ be a path prefix of length n and π be a path. Then, the *concatenation* of σ and π is given by

$$(\sigma \circ \pi)(i) = \begin{cases} \sigma(i) & : \quad i < n \\ \pi(i-n) & : \quad i \geq n \end{cases} \qquad \text{for all } i \in \mathbb{N}.$$

Note that we will only use the special case where the path prefix σ consists of a single state. For simplicity, we identify this single state with the path prefix itself, i.e., we identify state s with the path prefix $\sigma : 0 \mapsto s$.

Now let us start with the auxiliary results towards a counterexample path construction for queries in $\overline{\text{LTLQ}^x}$. As in the exactness proof of LTLQ^x, the tricky part in this proof is to find suitable auxiliary results.

Recall that every query in $\overline{\text{LTLQ}^x}$ has a subquery in LTLQ^1, which will be used as the starting point in our proof. At first, we need the following property, since LTLQ^1 and LTLQ^2 depend on each other (cf. Figure 1).

Lemma 4.19. *Let $\gamma \in LTLQ^1 \cup LTLQ^2$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi^{4n} \models \gamma[p]$, $\pi^{4n+2} \models \gamma[q]$, and $\pi^{2n} \not\models \gamma[p \wedge q]$ for all $n \in \mathbb{N}$.*

Proof. Structural induction on γ. See Appendix B for details. □

Since LTLQ^2 queries are the only subqueries of queries in LTLQ^1 (cf. Figure 1), the previous lemma can be used to show the following property.

Lemma 4.20. *Let $\gamma \in LTLQ^1$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi^{4n} \models \gamma[p]$ and $\pi^{4n+2} \models \gamma[q]$ for all $n \in \mathbb{N}$ as well as $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

Proof. Structural induction on γ. See Appendix B for details. □

Now we will prove a series of auxiliary results. These results will then be composed in order to obtain a counterexample path to the collecting property for all simple queries in $\overline{\text{LTLQ}^x}$.

The following lemma is an auxiliary result on LTLQ^5 and LTLQ^6. Since $\text{LTLQ}^5 \cup \text{LTLQ}^6$ and $\text{LTLQ}^3 \cup \text{LTLQ}^4$ depend on each other (cf. Figure 1), we have to make preliminary assumptions on subqueries in LTLQ^3 and LTLQ^4. For subqueries in LTLQ^1, we can directly use Lemma 4.20.

Lemma 4.21. *Let $\gamma \in LTLQ^5 \cup LTLQ^6$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^3$ and $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi^{4n} \models \gamma[p] \wedge \gamma[q]$ for all $n \in \mathbb{N}$ and $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

Proof. Structural induction on γ. See Appendix B for details. □

The following lemma is an auxiliary result on LTLQ^6 that will only be used in the proof of Lemma 4.23. Because of the additional assumption of a preceding global operator, we obtain a stronger result than in Lemma 4.21.

Lemma 4.22. *Let $\gamma = \mathbf{G}\, \gamma''$ be a simple LTL query where $\gamma'' \in LTLQ^6$. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma \models \mathbf{G}\, \gamma'[p] \wedge \mathbf{G}\, \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Suppose further that for every $LTLQ^5$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi \models \mathbf{G}\, \gamma[p] \wedge \mathbf{G}\, \gamma[q]$ and $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

Proof. Structural induction on γ. See Appendix B for details. □

The following lemma is an auxiliary result on LTLQ^3. Since LTLQ^3 and $\text{LTLQ}^4 \cup \text{LTLQ}^5 \cup \text{LTLQ}^6$ depend on each other (cf. Figure 1), we have to make preliminary assumptions on subqueries in LTLQ^4, LTLQ^5, and LTLQ^6. For subqueries in LTLQ^1 and subqueries in LTLQ^6 that are preceded by a global operator, we can directly use Lemma 4.20 and Lemma 4.22.

Lemma 4.23. *Let $\gamma \in LTLQ^3$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma \models \mathbf{G}\, \gamma'[p] \wedge \mathbf{G}\, \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Suppose further that for every $LTLQ^5$ and $LTLQ^6$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ for all $n \in \mathbb{N}$ and $\sigma \models \mathbf{G}\, \neg\gamma'[p \wedge q]$. Then, there exists a path π such that $\pi \models \mathbf{G}\, \gamma[p] \wedge \mathbf{G}\, \gamma[q]$ and $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

Proof. Structural induction on γ. See Appendix B for details. □

The following lemma is an auxiliary result on LTLQ^4. Since LTLQ^4 and $\text{LTLQ}^3 \cup \text{LTLQ}^5 \cup \text{LTLQ}^6$ depend on each other (cf. Figure 1), we have to make preliminary assumptions on subqueries in LTLQ^3, LTLQ^5, and LTLQ^6.

Lemma 4.24. *Let $\gamma \in LTLQ^4$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^3$, $LTLQ^5$, and $LTLQ^6$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi \models \mathbf{G}\, \gamma[p] \wedge \mathbf{G}\, \gamma[q]$ and $\pi^{4n} \not\models \gamma[p \wedge q]$ for all $n \in \mathbb{N}$.*

PROOF. Structural induction on γ. See Appendix B for details. □

Now we have finished our basic results. In the following, we will successively reduce the number of assumptions in the above lemmas; in particular, we will first remove the assumptions on LTLQ^4 subqueries. In order to obtain the statement of Lemma 4.23 without its assumption on LTLQ^4 subqueries, we use an inductive proof on the number of subqueries in LTLQ^4.

Lemma 4.25. *Let $\gamma \in LTLQ^3$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^5$ and $LTLQ^6$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ for all $n \in \mathbb{N}$ and $\sigma \models \mathbf{G}\, \neg\gamma'[p \wedge q]$. Then, there exists a path π such that $\pi \models \mathbf{G}\, \gamma[p] \wedge \mathbf{G}\, \gamma[q]$ and $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

PROOF. Induction on the number of subqueries in LTLQ^4.
Induction start: If γ contains no subquery in LTLQ^4, then the assumption of Lemma 4.23 on subqueries in LTLQ^4 is trivially satisfied. Thus, the only remaining assumptions of Lemma 4.23 are on queries in LTLQ^5 and LTLQ^6, which are satisfied by the current assumptions. Hence, we obtain the statement by Lemma 4.23.
Induction step: Let γ' be any LTLQ^4 subquery of γ. If γ' contains no subquery in LTLQ^3, then the assumption of Lemma 4.24 on subqueries in LTLQ^3 is trivially satisfied. Otherwise, let γ'' be any LTLQ^3 subquery of γ'. Since the number of LTLQ^4 subqueries of γ'' must be less than the number of LTLQ^4 subqueries of γ, we can apply the induction hypothesis and obtain the statement for γ''. Thus, since γ'' was chosen w.l.o.g., the assumption of Lemma 4.24 on subqueries in LTLQ^3 is again satisfied. Hence, the only remaining assumptions of Lemma 4.24 in both cases are on queries in LTLQ^5 and LTLQ^6, which are satisfied by the current assumptions. So in both cases we can apply Lemma 4.24 to γ'. Since γ' was chosen w.l.o.g., the assumption of Lemma 4.23 on subqueries in LTLQ^4 is satisfied. Thus, the only remaining assumptions of Lemma 4.23 are on queries in LTLQ^5 and LTLQ^6, which are satisfied by the current assumptions. Hence, we obtain the statement by Lemma 4.23. □

In order to obtain the statement of Lemma 4.21 without its assumption on LTLQ^4 subqueries, we use an inductive proof on the number of subqueries in LTLQ^4.

Lemma 4.26. *Let $\gamma \in LTLQ^5 \cup LTLQ^6$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^3$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi^{4n} \models \gamma[p] \wedge \gamma[q]$ for all $n \in \mathbb{N}$ and $\pi \models \mathbf{G}\, \neg\gamma[p \wedge q]$.*

PROOF. Induction on the number of subqueries in LTLQ^4.
Induction start: If γ contains no subquery in LTLQ^4, then the assumption of Lemma 4.21 on subqueries in LTLQ^4 is trivially satisfied. Thus, the only remaining assumption of Lemma 4.21 is on queries in LTLQ^3, which is satisfied by the current assumption. Hence, we obtain the statement by Lemma 4.21.
Induction step: Let γ' be any LTLQ^4 subquery of γ. If γ' contains no subquery in $\mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$, then the assumptions of Lemma 4.24 on subqueries in LTLQ^5 and LTLQ^6 are trivially satisfied. Otherwise, let γ'' be any LTLQ^5 or LTLQ^6 subquery of γ'. Since the number of LTLQ^4 subqueries of γ'' must be

less than the number of LTLQ^4 subqueries of γ, we can apply the induction hypothesis and obtain the statement for γ''. Thus, since γ'' was chosen w.l.o.g., the assumptions of Lemma 4.24 on subqueries in LTLQ^5 and LTLQ^6 are again satisfied. Hence, the only remaining assumption of Lemma 4.24 in both cases is on queries in LTLQ^3, which is satisfied by the current assumption. So in both cases we can apply Lemma 4.24 to γ'. Since γ' was chosen w.l.o.g., the assumption of Lemma 4.21 on subqueries in LTLQ^4 is satisfied. Thus, the only remaining assumption of Lemma 4.21 is on queries in LTLQ^3, which is satisfied by the current assumption. Hence, we obtain the statement by Lemma 4.21. □

Now we have obtained each result with assumptions on at most two kinds of subqueries. In the following we continue in the same manner as above. In order to obtain the statement of Lemma 4.26 without assumptions, we use an inductive proof on the number of subqueries in LTLQ^3.

Lemma 4.27. *Let $\gamma \in LTLQ^5 \cup LTLQ^6$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi^{4n} \models \gamma[p] \wedge \gamma[q]$ for all $n \in \mathbb{N}$ and $\pi \models \mathbf{G}\,\neg\gamma[p \wedge q]$.*

PROOF. Induction on the number of subqueries in LTLQ^3.
Induction start: If γ contains no subquery in LTLQ^3, then the assumption of Lemma 4.26 on subqueries in LTLQ^3 is trivially satisfied and we obtain the statement by Lemma 4.26.
Induction step: Let γ' be any LTLQ^3 subquery of γ. If γ' contains no subquery in LTLQ^5∪LTLQ^6, then the assumptions of Lemma 4.25 are trivially satisfied. Otherwise, let γ'' be any LTLQ^5 or LTLQ^6 subquery of γ'. Since the number of LTLQ^3 subqueries of γ'' must be less than the number of LTLQ^3 subqueries of γ, we can apply the induction hypothesis and obtain the statement for γ''. Thus, since γ'' was chosen w.l.o.g., the assumptions of Lemma 4.25 are again satisfied. So in both cases we can apply Lemma 4.25 to γ'. Since γ' was chosen w.l.o.g., the assumption of Lemma 4.26 is satisfied. Hence, we obtain the statement by Lemma 4.26. □

Since the assumptions of Lemma 4.25 are satisfied according to Lemma 4.27, we trivially obtain the following corollary by Lemma 4.25.

Corollary 4.28. *Let $\gamma \in LTLQ^3$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi \models \mathbf{G}\,\gamma[p] \wedge \mathbf{G}\,\gamma[q]$ and $\pi \models \mathbf{G}\,\neg\gamma[p \wedge q]$.*

Since the assumptions of Lemma 4.24 are satisfied according to Lemma 4.27 and Corollary 4.28, we trivially obtain the following corollary by Lemma 4.24.

Corollary 4.29. *Let $\gamma \in LTLQ^4$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi \models \mathbf{G}\,\gamma[p] \wedge \mathbf{G}\,\gamma[q]$ and $\pi^{4n} \not\models \gamma[p \wedge q]$ for all $n \in \mathbb{N}$.*

Let us now summarize our results so far: According to Lemma 4.27, Corollary 4.28, and Corollary 4.29, we know that for each simple query γ in LTLQ^3, LTLQ^4, LTLQ^5, and LTLQ^6 there exists a path π such that

$$\pi^{f_p^\gamma(n)} \models \gamma[p] \quad \text{and} \quad \pi^{f_q^\gamma(n)} \models \gamma[q], \quad \text{but} \quad \pi^{f_{p\wedge q}^\gamma(n)} \not\models \gamma[p \wedge q]$$

$f^\gamma_\varphi(n)$	$\varphi = p$	$\varphi = q$	$\varphi = p \wedge q$
$\gamma \in \text{LTLQ}^3$	n	n	n
$\gamma \in \text{LTLQ}^4$	n	n	$4n$
$\gamma \in \text{LTLQ}^5$	$4n$	$4n$	n
$\gamma \in \text{LTLQ}^6$	$4n$	$4n$	n

Table II. Structure of counterexample paths

where $f^\gamma_\varphi : \mathbb{N} \to \mathbb{N}$ is a function in n defined in Table II. Recall that $\overline{\text{LTLQ}^x}$ is defined as the union of LTLQ^3, LTLQ^4, LTLQ^5, and LTLQ^6 (see Definition 4.3). Thus, by Lemma 4.27, Corollary 4.28, and Corollary 4.29 we obtain the following property from the special case of $n = 0$.

Corollary 4.30. *Let $\gamma \in \overline{LTLQ^x}$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi \models \gamma[p] \wedge \gamma[q]$ but $\pi \not\models \gamma[p \wedge q]$, i.e., γ is not collecting.*

Finally, we obtain our second main result by Corollary 4.30 and Theorem 3.19.

Theorem 4.31. *Every simple query in $\overline{LTLQ^x}$ is not exact.*

Thus, we have shown that LTLQ^x is maximal in the sense that all templates not in LTLQ^x have simple instantiations that are not exact. Hence, LTLQ^x represents a syntactic characterization of exact LTL queries.

***Remark* 4.32.** As already mentioned, $\overline{\text{LTLQ}^x}$ contains also exact queries. However, these queries cannot be simple. In fact, whether a query in $\overline{\text{LTLQ}^x}$ is exact depends on the chosen instantiation. For example, the query $a\,\mathbf{U}\,(b \wedge (?\,\mathbf{U}\,c)) \in \text{LTLQ}^3$ is simple and therefore not exact. In contrast, the queries $a\,\mathbf{U}\,(b \wedge (?\,\mathbf{U}\,\mathbf{G}\,c)) \in \text{LTLQ}^3$ and $((b\,\mathbf{U}\,?) \wedge \mathbf{X}\,\neg a)\,\mathbf{U}\,a \in \text{LTLQ}^4$ are exact. Note, however, that it is not the case that all non-simple queries in $\overline{\text{LTLQ}^x}$ are exact. For example, the query $a\,\mathbf{U}\,(\mathbf{X}\,b \wedge (?\,\mathbf{U}\,c)) \in \text{LTLQ}^3$ is not simple and not exact.

***Remark* 4.33.** Finally, let us mention that our characterization also yields a characterization of LTL queries that are distributive over *disjunction*, i.e., of queries γ which satisfy $\gamma[\varphi \vee \psi] \Leftrightarrow \gamma[\varphi] \vee \gamma[\psi]$. This is the case because our language LTLQ^m of monotonic queries is closed under negation (cf. Section 2.2). In particular, for any LTL query γ let $\bar{\gamma}$ denote the dual query of γ, that is, $\bar{\gamma}$ is obtained from $\neg\gamma$ by building the NNF and removing negation in front of the placeholder. Clearly, it always holds that $\bar{\bar{\gamma}} = \gamma$. Now, let γ be distributive over conjunction. It is then not hard to see that $\bar{\gamma}$ is distributive over disjunction: $\bar{\gamma}[\varphi \vee \psi] \Leftrightarrow \neg\neg\bar{\gamma}[\varphi \vee \psi] \Leftrightarrow \neg\gamma[\neg\varphi \wedge \neg\psi] \Leftrightarrow \neg(\gamma[\neg\varphi] \wedge \gamma[\neg\psi]) \Leftrightarrow \neg\gamma[\neg\varphi] \vee \neg\gamma[\neg\psi] \Leftrightarrow \bar{\gamma}[\varphi] \vee \bar{\gamma}[\psi]$. Symmetrically it can be shown that $\bar{\gamma}$ is distributive over conjunction if γ is distributive over disjunction. Moreover, since LTLQ^m is closed under negation, we know that $\gamma \in \text{LTLQ}^m$ iff $\bar{\gamma} \in \text{LTLQ}^m$. Hence, the queries dual to the queries in LTLQ^x yield a syntactic characterization of LTL queries that are distributive over disjunction. However, note that in contrast to conjunction, LTL model checking of a disjunction cannot be split, i.e., $\mathfrak{K} \models \gamma[\varphi] \vee \gamma[\psi]$ is not equivalent to $\mathfrak{K} \models \gamma[\varphi]$ or $\mathfrak{K} \models \gamma[\psi]$.

5. CONCLUSION

In this paper, we have presented a syntactic characterization of distributive LTL specifications (resp. exact LTL queries) using query templates. For the proof of exactness, we have introduced stronger variants of the collecting property (strong, boundary, and intermediate collecting) which give us additional knowledge of the queries in the classes LTLQ^i. Thus, we do not only know that every query in LTLQ^x is exact, but we also know that every query in LTLQ^1 is collecting, every query in LTLQ^2 is intermediate collecting, and every query in LTLQ^7 is boundary collecting. We believe that this knowledge can be exploited to increase the degree of determinism when evaluating specifications or solving queries. Although not explicitly stated in his paper, Chan's symbolic algorithm [Chan 2000] for solving CTL queries is based on this principle; see [Samer and Veith 2005] for a discussion.

A syntactic characterization of exact CTL queries is still open. So far we have found an extension of our fragment in [Samer and Veith 2003], which is described by a context-free template grammar consisting of 10 non-terminals [Samer and Veith 2005], but this fragment is still not maximal.

A. PROOF OF EXACTNESS

Lemma 4.5. *Let $\gamma \in LTLQ^2$. Suppose that every subquery in $LTLQ^1$ is weak collecting. Then, γ is intermediate collecting.*

Proof. Structural induction on γ.

Induction start:

—Let $\gamma = \gamma' \,\mathbf{U}\, \theta$ such that $\gamma' \in \mathrm{LTLQ}^1$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain the statement by $\pi^k \models \gamma[\bot]$. Otherwise, if $k \geq n$, we know that $\pi^{[n,k)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma' \,\mathbf{W}\, \theta = (\mathbf{G}\, \gamma') \vee (\gamma' \,\mathbf{U}\, \theta)$ such that $\gamma' \in \mathrm{LTLQ}^1$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that $\pi \models \mathbf{G}\, \gamma'[\varphi]$ and $\pi^n \models \mathbf{G}\, \gamma'[\psi]$. Thus, it holds that $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^n \models \mathbf{G}\, \gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain the statement by $\pi^k \models \gamma[\bot]$. Otherwise, if $k \geq n$, we know that $\pi^{[n,k)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi^n \models \gamma'[\varphi \wedge \psi] \,\mathbf{U}\, \theta$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

Induction step:

—Let $\gamma = \theta \vee \gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. If $\pi \models \theta$ or $\pi^n \models \theta$, we trivially obtain $\pi \models \gamma[\bot]$ resp. $\pi^n \models \gamma[\varphi \wedge \psi]$. Otherwise, if $\pi \not\models \theta$ and $\pi^n \not\models \theta$, we know that $\pi \models \gamma'[\varphi]$ and $\pi^n \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^n \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $r < n$ and $\pi^r \models \gamma'[\bot]$, which trivially imply $\pi^n \models \gamma[\varphi \wedge \psi]$ and $\pi^r \models \gamma[\bot]$, respectively.

—Let $\gamma = \mathbf{X}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that $\pi^1 \models \gamma'[\varphi]$ and $\pi^{n+1} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{n+1} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $r < n$ and $\pi^{r+1} \models \gamma'[\bot]$, which imply $\pi^n \models \gamma[\varphi \wedge \psi]$ and $\pi^r \models \gamma[\bot]$, respectively.

—Let $\gamma = \gamma'\,\mathbf{U}\,\theta$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain the statement by $\pi^k \models \gamma[\bot]$. Otherwise, if $k \geq n$, we know that $\pi^{[n,k)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$ and $\pi^k \models \theta$. Hence, by induction hypothesis, we obtain $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{U}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $\min(k, n + l) \leq r < \max(k, n + l)$ and $\pi^r \models \gamma'[\bot]$. If $r < n$, we trivially obtain the statement by $\pi^r \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that either $\pi^{[n,r)} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$ or $\pi^{[n,\max(k,n+l))} \models \theta$ and $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. Thus, in both cases we have $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathbf{W}\,\theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\theta)$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that $\pi \models \mathbf{G}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$. Thus, it holds that $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^n \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain the statement by $\pi^k \models \gamma[\bot]$. Otherwise, if $k \geq n$, we know that $\pi^{[n,k)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,k)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi^n \models \gamma'[\varphi \wedge \psi]\,\mathbf{U}\,\theta$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,\gamma')$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^n \models \mathbf{G}\,\theta$, we trivially obtain $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $\min(k, n+l) \leq r < \max(k, n+l)$ and $\pi^r \models \gamma'[\bot]$. If $r < n$, we trivially obtain the statement by $\pi^r \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that either $\pi^{[n,r)} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$ or $\pi^{[n,\max(k,n+l))} \models \theta$ and $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. Thus, in both cases we have $\pi^n \models \theta\,\mathbf{U}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

This concludes the proof. □

Lemma 4.6. *Let $\gamma \in LTLQ^1$. Suppose that every subquery in $LTLQ^2$ is intermediate collecting. Then, γ is weak collecting.*

Proof. Structural induction on γ.

Induction start:

—If γ is the placeholder, then γ is trivially weak collecting.

—Let $\gamma = \theta \wedge \gamma'$ such that $\gamma' \in \text{LTLQ}^2$. It is easy to see that $\gamma[\varphi] \wedge \gamma[\psi]$ is equivalent to $\theta \wedge (\gamma'[\varphi] \wedge \gamma'[\psi])$. Hence, by assumption, we obtain $\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma' \, \mathring{\mathbf{U}} \, \theta = \gamma' \, \mathbf{U} \, (\gamma' \wedge \theta)$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and therefore $\pi^{[0,k]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \, \mathring{\mathbf{U}} \, \gamma' = \theta \, \mathbf{U} \, (\theta \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^l \models \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{\max(k,l)} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $\min(k,l) \leq r < \max(k,l)$ and $\pi^r \models \gamma'[\bot]$. Consequently, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that either $\pi^{[0,r]} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$ or $\pi^{[0,\max(k,l)]} \models \theta$ and $\pi^{\max(k,l)} \models \gamma'[\varphi \wedge \psi]$. Thus, in both cases we have $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \, \bar{\mathbf{U}} \, \gamma' = \theta \, \mathbf{U} \, (\neg\theta \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \neg\theta$ and therefore $\pi^k \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^k \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma' \, \mathring{\mathbf{W}} \, \theta = (\mathbf{G} \, \gamma') \vee (\gamma' \, \mathbf{U} \, (\gamma' \wedge \theta))$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that $\pi^{[0,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi \models \mathbf{G} \, \gamma'[\varphi \wedge \psi]$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and therefore $\pi^{[0,k]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma'[\varphi \wedge \psi] \, \mathring{\mathbf{U}} \, \theta$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \, \mathring{\mathbf{W}} \, \gamma' = (\mathbf{G} \, \theta) \vee (\theta \, \mathbf{U} \, (\theta \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Now, we have to distinguish between two cases: (i) If $\pi \models \mathbf{G} \, \theta$, we trivially obtain $\pi \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^l \models \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{\max(k,l)} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $\min(k,l) \leq r < \max(k,l)$ and $\pi^r \models \gamma'[\bot]$. Consequently, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that either $\pi^{[0,r]} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$ or $\pi^{[0,\max(k,l)]} \models \theta$ and $\pi^{\max(k,l)} \models \gamma'[\varphi \wedge \psi]$. Thus, in both cases we have $\pi \models \theta \, \mathring{\mathbf{U}} \, \gamma'[\varphi \wedge \psi]$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \, \bar{\mathbf{W}} \, \gamma' = (\mathbf{G} \, \theta) \vee (\theta \, \mathbf{U} \, (\neg\theta \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Now, we have to distinguish between two cases: (i) If $\pi \models \mathbf{G} \, \theta$, we trivially obtain $\pi \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \neg\theta$ and therefore $\pi^k \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by assumption, we obtain $\pi^k \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \theta \, \bar{\mathbf{U}} \, \gamma'[\varphi \wedge \psi]$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$.

Induction step:

—Let $\gamma = \theta \wedge \gamma'$. It is easy to see that $\gamma[\varphi] \wedge \gamma[\psi]$ is equivalent to $\theta \wedge (\gamma'[\varphi] \wedge \gamma'[\psi])$. Hence, by induction hypothesis, we obtain $\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \vee \gamma'$. It is easy to see that $\gamma[\varphi] \wedge \gamma[\psi]$ is equivalent to $\theta \vee (\gamma'[\varphi] \wedge \gamma'[\psi])$. Hence, by induction hypothesis, we obtain $\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \mathbf{X}\,\gamma'$. It is easy to see that $\gamma[\varphi] \wedge \gamma[\psi]$ is equivalent to $\mathbf{X}\,(\gamma'[\varphi] \wedge \gamma'[\psi])$. Hence, by induction hypothesis, we obtain $\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\theta = \gamma'\,\mathbf{U}\,(\gamma' \wedge \theta)$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and therefore $\pi^{[0,k]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\bar{\mathbf{U}}\,\gamma' = \theta\,\mathbf{U}\,(\neg\theta \wedge \gamma')$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Then, we know that there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \neg\theta$ and therefore $\pi^k \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^k \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \theta))$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that $\pi^{[0,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and therefore $\pi^{[0,k]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma'[\varphi \wedge \psi]\,\mathring{\mathbf{U}}\,\theta$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\bar{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,(\neg\theta \wedge \gamma'))$. Suppose that $\pi \models \gamma[\varphi] \wedge \gamma[\psi]$. Now, we have to distinguish between two cases: (i) If $\pi \models \mathbf{G}\,\theta$, we trivially obtain $\pi \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \neg\theta$ and therefore $\pi^k \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^k \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \theta\,\bar{\mathbf{U}}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi \models \gamma[\varphi \wedge \psi]$.

This concludes the proof. ☐

Lemma 4.9. *Let $\gamma \in LTLQ^4$. Suppose that for every subquery $\gamma' \in LTLQ^5$ it holds that $\mathbf{G}\,\gamma'$ is weak collecting. Then, $\mathbf{F}\,\gamma$ is boundary collecting.*

PROOF. Structural induction on γ.

Induction start:

—Let $\gamma = \gamma'\,\mathbf{U}\,\theta$ such that $\gamma' \in \mathrm{LTLQ}^3 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Since θ trivially implies $\gamma[\varphi \wedge \psi]$, we obtain $\pi^{n+l} \models \gamma[\varphi \wedge \psi]$ and therefore $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathbf{W}\,\theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\theta)$ such that $\gamma' \in \mathrm{LTLQ}^5$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \mathbf{G}\,\gamma'[\varphi]$ and $\pi^{n+l} \models \mathbf{G}\,\gamma'[\psi]$, which implies $\pi^{\max(k,n+l)} \models \mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi]$. Hence, by assumption, we obtain $\pi^{\max(k,n+l)} \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^{\max(k,n+l)} \models \gamma[\varphi \wedge \psi]$ and therefore $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exists $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain $\pi^k \models \gamma[\bot]$ and therefore $\pi \models \mathbf{F}\,\gamma[\bot]$. Otherwise, if $k \geq n$, we trivially obtain $\pi^k \models \gamma[\varphi \wedge \psi]$ and therefore $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$.

Induction step:

—Let $\gamma = \theta \vee \gamma'$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. So we have $\pi \models \mathbf{F}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \mathbf{F}\,\gamma'[\bot]$ or $\pi^n \models \mathbf{F}\,\gamma'[\varphi \wedge \psi]$, which trivially imply $\pi \models \mathbf{F}\,\gamma[\bot]$ and $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$, respectively. (ii) Otherwise, there exists $k \in \mathbb{N}$ such that $\pi^k \models \theta$. If $k < n$, we trivially obtain $\pi^k \models \gamma[\bot]$ and therefore $\pi \models \mathbf{F}\,\gamma[\bot]$. Otherwise, if $k \geq n$, we trivially obtain $\pi^k \models \gamma[\varphi \wedge \psi]$ and therefore $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \mathbf{X}\,\gamma'$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^{k+1} \models \gamma'[\varphi]$ and $\pi^{n+l+1} \models \gamma'[\psi]$. So we have $\pi^1 \models \mathbf{F}\,\gamma'[\varphi]$ and $\pi^{n+1} \models \mathbf{F}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^1 \models \mathbf{F}\,\gamma'[\bot]$ or $\pi^{n+1} \models \mathbf{F}\,\gamma'[\varphi \wedge \psi]$, which imply $\pi \models \mathbf{F}\,\gamma[\bot]$ and $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \gamma'\,\mathbf{U}\,\theta$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Since θ trivially implies $\gamma[\varphi \wedge \psi]$, we obtain $\pi^{n+l} \models \gamma[\varphi \wedge \psi]$ and therefore $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{U}\,\gamma'$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. So we have $\pi \models \mathbf{F}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \mathbf{F}\,\gamma'[\bot]$ or $\pi^n \models \mathbf{F}\,\gamma'[\varphi \wedge \psi]$, which imply $\pi \models \mathbf{F}\,\gamma[\bot]$ and $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \theta\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,\gamma')$. Suppose that $\pi \models \mathbf{F}\,\gamma[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If there exists $k \in \mathbb{N}$ such that $\pi^k \models \mathbf{G}\,\theta$, we trivially obtain $\pi^k \models \gamma[\bot]$ and therefore $\pi \models \mathbf{F}\,\gamma[\bot]$. (ii) Otherwise, there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. So we have $\pi \models \mathbf{F}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \mathbf{F}\,\gamma'[\bot]$ or $\pi^n \models \mathbf{F}\,\gamma'[\varphi \wedge \psi]$, which imply $\pi \models \mathbf{F}\,\gamma[\bot]$ and $\pi^n \models \mathbf{F}\,\gamma[\varphi \wedge \psi]$, respectively.

This concludes the proof. □

Lemma 4.10. *Let $\gamma \in LTLQ^5$. Suppose that for every subquery $\gamma' \in LTLQ^4$ it holds that $\mathbf{F}\,\gamma'$ is weak collecting. Then, $\mathbf{G}\,\gamma$ is weak collecting.*

Proof. Structural induction on γ.

Induction start:

—Let $\gamma = \theta\,\mathring{\mathbf{U}}\,\gamma' = \theta\,\mathbf{U}\,(\theta \wedge \gamma')$ such that $\gamma' \in \mathrm{LTLQ}^4$. Consider the formula $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$, which is equivalent to $\mathbf{G}\,(\gamma[\varphi] \wedge \gamma[\psi])$. Since $\gamma[\varphi] \wedge \gamma[\psi]$ implies $\mathbf{F}\,\gamma'[\varphi] \wedge \mathbf{F}\,\gamma'[\psi]$, we obtain by assumption $\mathbf{GF}\,\gamma'[\varphi \wedge \psi]$. On the other hand, it is easy to see that $\mathbf{G}\,\gamma[\varphi]$ implies $\mathbf{G}\,\theta$. Thus, $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ implies $\mathbf{G}\,\theta \wedge \mathbf{GF}\,\gamma'[\varphi \wedge \psi]$, which is equivalent to $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,\gamma')$ such that $\gamma' \in \mathrm{LTLQ}^1$. Consider the formula $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$, which is equivalent to $\mathbf{G}\,(\gamma[\varphi] \wedge \gamma[\psi])$. If θ does not hold, we know that $\gamma[\varphi] \wedge \gamma[\psi]$ implies $\gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by Lemma 4.7, we obtain $\gamma'[\varphi \wedge \psi]$. Thus, $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ implies $\mathbf{G}\,(\theta \vee \gamma'[\varphi \wedge \psi])$, which is equivalent to $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \mathring{\mathbf{W}} \gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,(\theta \wedge \gamma'))$ such that $\gamma' \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^3 \cup \mathrm{LTLQ}^4 \cup \mathrm{LTLQ}^6$. Consider the formula $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$. It is easy to see that $\mathbf{G}\,\gamma[\varphi]$ implies $\mathbf{G}\,\theta$. Thus, since $\mathbf{G}\,\theta$ is equivalent to $\mathbf{GG}\,\theta$ and $\mathbf{G}\,\theta$ trivially implies $\gamma[\varphi \wedge \psi]$, we have $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

Induction step:

—Let $\gamma = \theta \wedge \gamma'$. It is easy to see that $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ is equivalent to $\mathbf{G}\,\theta \wedge (\mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi])$. Hence, by induction hypothesis, we obtain $\mathbf{G}\,\theta \wedge \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which is equivalent to $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \mathbf{X}\,\gamma'$. It is easy to see that $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ is equivalent to $\mathbf{XG}\,\gamma'[\varphi] \wedge \mathbf{XG}\,\gamma'[\psi]$, which in turn is equivalent to $\mathbf{X}\,(\mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi])$. Hence, by induction hypothesis, we obtain $\mathbf{XG}\,\gamma'[\varphi \wedge \psi]$, which is equivalent to $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma' \mathring{\mathbf{U}} \theta = \gamma'\,\mathbf{U}\,(\gamma' \wedge \theta)$. It is easy to see that $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ implies $\mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\mathbf{G}\,\gamma'[\varphi \wedge \psi]$. On the other hand, it is easy to see that $\mathbf{G}\,\gamma[\varphi]$ implies $\mathbf{GF}\,\theta$. Thus, we know that $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ implies $\mathbf{G}\,\gamma'[\varphi \wedge \psi] \wedge \mathbf{GF}\,\theta$, which is equivalent to $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma' \mathring{\mathbf{W}} \theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \theta))$. It is easy to see that $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$ implies $\mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\mathbf{G}\,\gamma'[\varphi \wedge \psi]$. Thus, since $\mathbf{G}\,\gamma'[\varphi \wedge \psi]$ is equivalent to $\mathbf{GG}\,\gamma'[\varphi \wedge \psi]$ and $\mathbf{G}\,\gamma'[\varphi \wedge \psi]$ trivially implies $\gamma[\varphi \wedge \psi]$, we have $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta \mathring{\mathbf{W}} \gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,(\theta \wedge \gamma'))$. Consider the formula $\mathbf{G}\,\gamma[\varphi] \wedge \mathbf{G}\,\gamma[\psi]$. It is easy to see that $\mathbf{G}\,\gamma[\varphi]$ implies $\mathbf{G}\,\theta$. Thus, since $\mathbf{G}\,\theta$ is equivalent to $\mathbf{GG}\,\theta$ and $\mathbf{G}\,\theta$ trivially implies $\gamma[\varphi \wedge \psi]$, we have $\mathbf{G}\,\gamma[\varphi \wedge \psi]$.

This concludes the proof. □

Lemma 4.13. *Every query in LTLQ^7 is boundary collecting.*

PROOF. Structural induction on γ.

Induction start:

—Let $\gamma = \mathbf{F}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^2$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by Corollary 4.8, we obtain $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$ or an $r \in \mathbb{N}$ such that $\min(k, n+l) \le r < \max(k, n+l)$ and $\pi^r \models \gamma'[\bot]$. So, if $\pi^r \models \gamma'[\bot]$, we trivially have $\pi \models \gamma[\bot]$, and if $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$, we trivially have $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \mathbf{F}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^4$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$, that is, $\pi \models \mathbf{F}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{F}\,\gamma'[\psi]$. Hence, by Lemma 4.11, we obtain $\pi \models \mathbf{F}\,\gamma'[\bot]$ or $\pi^n \models \mathbf{F}\,\gamma'[\varphi \wedge \psi]$, that is, $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \mathbf{G}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^2 \cup \mathrm{LTLQ}^5$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Thus, we know that $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$, that is, $\pi^n \models \mathbf{G}\,\gamma'[\varphi] \wedge \mathbf{G}\,\gamma'[\psi]$. Hence, by Lemma 4.7 resp. Corollary 4.8 resp. Corollary 4.12, we obtain $\pi^{[n,\infty)} \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

Induction step:

—Let $\gamma = \theta \wedge \gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Thus, we know that $\pi \models \gamma'[\varphi]$ and $\pi^n \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \gamma'[\bot]$ or $\pi^n \models \gamma'[\varphi \wedge \psi]$. So we have $\pi \models \theta \wedge \gamma'[\bot]$ or $\pi^n \models \theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \theta \vee \gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. If $\pi \models \theta$ or $\pi^n \models \theta$, we trivially obtain $\pi \models \gamma[\bot]$ resp. $\pi^n \models \gamma[\varphi \wedge \psi]$. Otherwise, if $\pi \not\models \theta$ and $\pi^n \not\models \theta$, we know that $\pi \models \gamma'[\varphi]$ and $\pi^n \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi \models \gamma'[\bot]$ or $\pi^n \models \gamma'[\varphi \wedge \psi]$, which trivially imply $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \mathbf{X}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that $\pi^1 \models \gamma'[\varphi]$ and $\pi^{n+1} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^1 \models \gamma'[\bot]$ or $\pi^{n+1} \models \gamma'[\varphi \wedge \psi]$, that is, $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively.

—Let $\gamma = \mathbf{F}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. So, if $\pi^{\min(k,n+l)} \models \gamma'[\bot]$, we trivially have $\pi \models \gamma[\bot]$, and if $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$, we trivially have $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \mathbf{G}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Thus, we have $\pi \models \mathbf{G}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$, which implies $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^{[n,\infty)} \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathbf{U}\,\theta$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Thus, we have $\pi^{[0,k)} \models \gamma'[\varphi]$ and $\pi^{[n,n+l)} \models \gamma'[\psi]$. If $k < n$ and it does not hold that $\pi^{[0,k)} \models \gamma'[\bot]$ or $\pi^{[n,n+l)} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 < k$ and $j_0 < l$ such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k)} \models \gamma'[\bot]$ and $\pi^k \models \theta$ or $\pi^{[n,n+l)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{n+l} \models \theta$, that is, $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. Otherwise, if $k \geq n$, we know that $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{\min(k,n+l)} \models \theta$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\theta = \gamma'\,\mathbf{U}\,(\gamma' \wedge \theta)$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Thus, we have $\pi^{[0,k]} \models \gamma'[\varphi]$ and $\pi^{[n,n+l]} \models \gamma'[\psi]$. If $k < n$ and it does not hold that $\pi^{[0,k]} \models \gamma'[\bot]$ or $\pi^{[n,n+l]} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 \leq k$ and $j_0 \leq l$ such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k]} \models \gamma'[\bot]$ and $\pi^k \models \theta$ or $\pi^{[n,n+l]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{n+l} \models \theta$, that is, $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. Otherwise, if $k \geq n$, we know that $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{\min(k,n+l)} \models \theta$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{U}\,\gamma'$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that there exists $r \in \{k, n+l\}$ such that $\pi^{[n,r)} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathring{\mathbf{U}}\,\gamma' = \theta\,\mathbf{U}\,(\theta \wedge \gamma')$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k]} \models \theta$ and $\pi^k \models \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that there exists $r \in \{k, n+l\}$ such that $\pi^{[n,r]} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\bar{\mathbf{U}}\,\gamma' = \theta\,\mathbf{U}\,(\neg\theta \wedge \gamma')$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Then, we know that there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \neg\theta \wedge \gamma'[\varphi]$ and $\pi^{n+l} \models \neg\theta \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that $\pi^{[n,n+l)} \models \theta$ and $\pi^{n+l} \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathbf{W}\,\theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\theta)$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between three cases: (i) If $\pi^{[0,\infty)} \not\models \theta$, we know that $\pi \models \mathbf{G}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$, which implies $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^n \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, if there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{[n,\infty)} \not\models \theta$, we know that $k < n$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$. Thus, we have $\pi^{[0,k)} \models \gamma'[\varphi]$ and $\pi^{[n,\infty)} \models \gamma'[\psi]$. If it does not hold that $\pi^{[0,k)} \models \gamma'[\bot]$ or $\pi^{[n,\infty)} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 < k$ and j_0 such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k)} \models \gamma'[\bot]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma'[\bot]\,\mathbf{U}\,\theta$, or $\pi^n \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially imply $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. (iii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Thus, we have $\pi^{[0,k)} \models \gamma'[\varphi]$ and $\pi^{[n,n+l)} \models \gamma'[\psi]$. If $k < n$ and it does not hold that $\pi^{[0,k)} \models \gamma'[\bot]$ or $\pi^{[n,n+l)} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 < k$ and $j_0 < l$ such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k)} \models \gamma'[\bot]$ and $\pi^k \models \theta$ or $\pi^{[n,n+l)} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{n+l} \models \theta$, that is, $\pi \models \gamma'[\bot]\,\mathbf{U}\,\theta$ and $\pi^n \models \gamma'[\varphi \wedge \psi]\,\mathbf{U}\,\theta$, respectively, which trivially imply $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. Otherwise, if $k \geq n$, we know that $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,\min(k,n+l))} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{\min(k,n+l)} \models \theta$, that is, $\pi^n \models \gamma'[\varphi \wedge \psi]\,\mathbf{U}\,\theta$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\theta = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \theta))$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between three cases: (i) If

$\pi^{[0,\infty)} \not\models \theta$, we know that $\pi \models \mathbf{G}\,\gamma'[\varphi]$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$, which implies $\pi^{[n,\infty)} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^n \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, if there exists a least $k \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{[n,\infty)} \not\models \theta$, we know that $k < n$ and $\pi^n \models \mathbf{G}\,\gamma'[\psi]$. Thus, we have $\pi^{[0,k]} \models \gamma'[\varphi]$ and $\pi^{[n,\infty)} \models \gamma'[\psi]$. If it does not hold that $\pi^{[0,k]} \models \gamma'[\bot]$ or $\pi^{[n,\infty)} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 \leq k$ and j_0 such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k]} \models \gamma'[\bot]$ and $\pi^k \models \theta$, that is, $\pi \models \gamma'[\bot]\,\mathring{\mathbf{U}}\,\theta$, or $\pi^n \models \mathbf{G}\,\gamma'[\varphi \wedge \psi]$, which trivially imply $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. (iii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \theta$ and $\pi^{n+l} \models \theta$. Thus, we have $\pi^{[0,k]} \models \gamma'[\varphi]$ and $\pi^{[n,n+l]} \models \gamma'[\psi]$. If $k < n$ and it does not hold that $\pi^{[0,k]} \models \gamma'[\bot]$ or $\pi^{[n,n+l]} \models \gamma'[\varphi \wedge \psi]$, there must exist $i_0 \leq k$ and $j_0 \leq l$ such that $\pi^{i_0} \not\models \gamma'[\bot]$ and $\pi^{n+j_0} \not\models \gamma'[\varphi \wedge \psi]$. However, since $\pi^{i_0} \models \gamma'[\varphi]$ and $\pi^{n+j_0} \models \gamma'[\psi]$, this contradicts the induction hypothesis. Hence, we have either $\pi^{[0,k]} \models \gamma'[\bot]$ and $\pi^k \models \theta$ or $\pi^{[n,n+l]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{n+l} \models \theta$, that is, $\pi \models \gamma'[\bot]\,\mathring{\mathbf{U}}\,\theta$ and $\pi^n \models \gamma'[\varphi \wedge \psi]\,\mathring{\mathbf{U}}\,\theta$, respectively, which trivially imply $\pi \models \gamma[\bot]$ and $\pi^n \models \gamma[\varphi \wedge \psi]$, respectively. Otherwise, if $k \geq n$, we know that $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi] \wedge \gamma'[\psi]$. Hence, by induction hypothesis and Lemma 3.5, we obtain $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi \wedge \psi]$. So we have $\pi^{[n,\min(k,n+l)]} \models \gamma'[\varphi \wedge \psi]$ and $\pi^{\min(k,n+l)} \models \theta$, that is, $\pi^n \models \gamma'[\varphi \wedge \psi]\,\mathring{\mathbf{U}}\,\theta$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,\gamma')$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^n \models \mathbf{G}\,\theta$, we trivially obtain $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that there exists $r \in \{k, n+l\}$ such that $\pi^{[n,r)} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \theta\,\mathbf{U}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\mathring{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,(\theta \wedge \gamma'))$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^n \models \mathbf{G}\,\theta$, we trivially obtain $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \gamma'[\varphi]$ and $\pi^{n+l} \models \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k]} \models \theta$ and $\pi^k \models \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi \wedge \psi]$ by Lemma 3.5, we know that there exists $r \in \{k, n+l\}$ such that $\pi^{[n,r]} \models \theta$ and $\pi^r \models \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \theta\,\mathring{\mathbf{U}}\,\gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

—Let $\gamma = \theta\,\bar{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\theta) \vee (\theta\,\mathbf{U}\,(\neg\theta \wedge \gamma'))$. Suppose that $\pi \models \gamma[\varphi]$ and $\pi^n \models \gamma[\psi]$ for some $n \in \mathbb{N}$. Now, we have to distinguish between two cases: (i) If $\pi^n \models \mathbf{G}\,\theta$, we trivially obtain $\pi^n \models \gamma[\varphi \wedge \psi]$. (ii) Otherwise, there exist least $k, l \in \mathbb{N}$ such that $\pi^k \models \neg\theta \wedge \gamma'[\varphi]$ and $\pi^{n+l} \models \neg\theta \wedge \gamma'[\psi]$. Hence, by induction hypothesis, we obtain $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ or $\pi^{\max(k,n+l)} \models \gamma'[\varphi \wedge \psi]$. If $\pi^{\min(k,n+l)} \models \gamma'[\bot]$ and $k < n$, we have $\pi^{[0,k)} \models \theta$ and $\pi^k \models \neg\theta \wedge \gamma'[\bot]$, that is, $\pi \models \gamma[\bot]$. Otherwise, since $\gamma'[\bot]$ implies $\gamma'[\varphi\wedge\psi]$ by Lemma 3.5, we know that $\pi^{[n,n+l)} \models \theta$ and $\pi^{n+l} \models \neg\theta \wedge \gamma'[\varphi \wedge \psi]$, that is, $\pi^n \models \theta \,\bar{\mathbf{U}}\, \gamma'[\varphi \wedge \psi]$, which trivially implies $\pi^n \models \gamma[\varphi \wedge \psi]$.

This concludes the proof. $\square$

B. PROOF OF MAXIMALITY

Lemma 4.19. *Let $\gamma \in LTLQ^1 \cup LTLQ^2$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi^{4n} \models \gamma[p]$, $\pi^{4n+2} \models \gamma[q]$, and $\pi^{2n} \not\models \gamma[p \wedge q]$ for all $n \in \mathbb{N}$.*

PROOF. Structural induction on γ.

Induction start:

—If γ is the placeholder, the statement holds for path π iff for all $n \in \mathbb{N}$ it holds that $p \in \ell(\pi^{4n})$, $q \in \ell(\pi^{4n+2})$, and $\{p, q\} \not\subseteq \ell(\pi^{2n})$.

Induction step:

—Let $\gamma = \alpha \wedge \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi^{2n} \models \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\neg\varphi$ implies $\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \vee \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n+2} \models \gamma'[q]$, and $\pi^{2n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since φ implies $\alpha \vee \varphi$ and $\neg\alpha \wedge \neg\varphi$ is equivalent to $\neg(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{X}\,\gamma'$. By induction hypothesis, we obtain a path σ for γ'. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi^{4n} \models \mathbf{X}\,\gamma'[p]$, $\pi^{4n+2} \models \mathbf{X}\,\gamma'[q]$, and $\pi^{2n} \models \mathbf{X}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{X}\,\neg\varphi$ is equivalent to $\neg\mathbf{X}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathbf{U}\,\alpha$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ and $\alpha \in \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \mathbf{X}\,\alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \mathbf{X}\,\alpha$, and $\pi^{2n} \models \neg\gamma'[p \wedge q] \wedge \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \mathbf{X}\,\alpha$ implies $\varphi\,\mathbf{U}\,\alpha$ and $\neg\varphi \wedge \neg\alpha$ implies $\neg(\varphi\,\mathbf{U}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\alpha = \gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi^{2n} \models \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi\,\mathring{\mathbf{U}}\,\alpha$ and $\neg\varphi$ implies $\neg(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathbf{U}\,\gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n+2} \models \gamma'[q]$, and $\pi^{2n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since φ implies $\alpha\,\mathbf{U}\,\varphi$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha\,\mathbf{U}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathring{\mathbf{U}} \gamma' = \alpha \mathbf{U} (\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi^{2n} \models \neg\gamma'[p \wedge q] \wedge \mathbf{X} \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha \mathring{\mathbf{U}} \varphi$ and $\neg\varphi \wedge \mathbf{X} \neg\alpha$ implies $\neg(\alpha \mathring{\mathbf{U}} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \bar{\mathbf{U}} \gamma' = \alpha \mathbf{U} (\neg\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi^{2n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha \bar{\mathbf{U}} \varphi$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha \bar{\mathbf{U}} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \mathbf{W} \alpha = (\mathbf{G} \gamma') \vee (\gamma' \mathbf{U} \alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ and $\alpha \in \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \mathbf{X} \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \mathbf{X} \alpha$, and $\pi^{2n} \models \neg\gamma'[p \wedge q] \wedge \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \mathbf{X} \alpha$ implies $\varphi \mathbf{W} \alpha$ and $\neg\varphi \wedge \neg\alpha$ implies $\neg(\varphi \mathbf{W} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \mathring{\mathbf{W}} \alpha = (\mathbf{G} \gamma') \vee (\gamma' \mathbf{U} (\gamma' \wedge \alpha))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi^{2n} \models \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi \mathring{\mathbf{W}} \alpha$ and $\neg\varphi$ implies $\neg(\varphi \mathring{\mathbf{W}} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathbf{W} \gamma' = (\mathbf{G} \alpha) \vee (\alpha \mathbf{U} \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n+2} \models \gamma'[q]$, and $\pi^{2n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since φ implies $\alpha \mathbf{W} \varphi$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha \mathbf{W} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathring{\mathbf{W}} \gamma' = (\mathbf{G} \alpha) \vee (\alpha \mathbf{U} (\alpha \wedge \gamma'))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi^{2n} \models \neg\gamma'[p \wedge q] \wedge \mathbf{X} \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha \mathring{\mathbf{W}} \varphi$ and $\neg\varphi \wedge \mathbf{X} \neg\alpha$ implies $\neg(\alpha \mathring{\mathbf{W}} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \bar{\mathbf{W}} \gamma' = (\mathbf{G} \alpha) \vee (\alpha \mathbf{U} (\neg\alpha \wedge \gamma'))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi^{2n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha \bar{\mathbf{W}} \varphi$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha \bar{\mathbf{W}} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. □

Lemma 4.20. *Let $\gamma \in LTLQ^1$ be simple. Further, let p and q be atomic propositions not occurring in γ. Then, there exists a path π such that $\pi^{4n} \models \gamma[p]$ and $\pi^{4n+2} \models \gamma[q]$ for all $n \in \mathbb{N}$ as well as $\pi \models \mathbf{G} \neg\gamma[p \wedge q]$.*

PROOF. Structural induction on γ.

Induction start:

—If γ is the placeholder, the statement holds for path π iff for all $n \in \mathbb{N}$ it holds that $p \in \ell(\pi^{4n})$, $q \in \ell(\pi^{4n+2})$, and $\{p,q\} \not\subseteq \ell(\pi^n)$.

—Let $\gamma = \alpha \wedge \gamma'$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\alpha = \gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha)$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{G}\,(\neg\gamma'[p \wedge q] \vee \neg\alpha)$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi\,\mathring{\mathbf{U}}\,\alpha$ and $\mathbf{G}\,(\neg\varphi \vee \neg\alpha)$ is equivalent to $\mathbf{G}\,\neg(\varphi \wedge \alpha)$ which implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\alpha \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha\,\mathring{\mathbf{U}}\,\varphi$ and $\mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ and $\alpha \in \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,(\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha\,\bar{\mathbf{U}}\,\varphi$ and $\mathbf{G}\,(\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\neg\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha))$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{GF}\,\neg\gamma'[p \wedge q] \wedge \mathbf{G}\,(\neg\gamma'[p \wedge q] \vee \neg\alpha)$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi\,\mathring{\mathbf{W}}\,\alpha$ and $\mathbf{GF}\,\neg\varphi \wedge \mathbf{G}\,(\neg\varphi \vee \neg\alpha)$ is equivalent to $\mathbf{GF}\,\neg\varphi \wedge \mathbf{G}\,\neg(\varphi \wedge \alpha)$ which implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\alpha \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ and $\alpha \notin \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha\,\mathring{\mathbf{W}}\,\varphi$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg(\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^2$. By Lemma 4.19, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ and $\alpha \in \ell(\pi^{2n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha\,\bar{\mathbf{W}}\,\varphi$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg(\neg\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

Induction step:

—Let $\gamma = \alpha \wedge \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \vee \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n+2} \models \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \wedge \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since φ implies $\alpha \vee \varphi$ and $\mathbf{G}\,(\neg\alpha \wedge \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{X}\,\gamma'$. By induction hypothesis, we obtain a path σ for γ'. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi^{4n} \models \mathbf{X}\,\gamma'[p]$, $\pi^{4n+2} \models \mathbf{X}\,\gamma'[q]$, and $\pi \models \mathbf{XG}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{XG}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{X}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\alpha = \gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi\,\mathring{\mathbf{U}}\,\alpha$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha\,\bar{\mathbf{U}}\,\varphi$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{2n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n+2} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi\,\mathring{\mathbf{W}}\,\alpha$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma'))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \neg\alpha \wedge \gamma'[p]$, $\pi^{4n+2} \models \neg\alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\neg\alpha \wedge \varphi$ implies $\alpha\,\bar{\mathbf{W}}\,\varphi$ and $\mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. ☐

Lemma 4.21. *Let $\gamma \in LTLQ^5 \cup LTLQ^6$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^3$ and $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi^{4n} \models \gamma[p] \wedge \gamma[q]$ for all $n \in \mathbb{N}$ and $\pi \models \mathbf{G}\,\neg\gamma[p \wedge q]$.*

Proof. Structural induction on γ.

Induction start:

—Let $\gamma = \alpha \wedge \gamma'$ such that $\gamma' \in \text{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ and $\alpha \notin \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\alpha \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ and $\alpha \notin \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha\,\mathring{\mathbf{U}}\,\varphi$ and $\mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,\gamma')$ such that $\gamma' \in \text{LTLQ}^1$. By Lemma 4.20, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{[4n,4n+2)})$ and $\alpha \notin \ell(\pi^{4n+2})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \mathbf{X}\,\alpha \wedge \mathbf{XX}\,\gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since both φ and $\alpha \wedge \mathbf{X}\,\alpha \wedge \mathbf{XX}\,\varphi$ imply $\alpha\,\mathbf{W}\,\varphi$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathbf{W}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\alpha \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^1$. By Lemma 4.20, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{[4n,4n+2]})$ and $\alpha \notin \ell(\pi^{4n+3})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \mathbf{X}\,\alpha \wedge \mathbf{XX}\,(\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since both $\alpha \wedge \varphi$ and $\alpha \wedge \mathbf{X}\,\alpha \wedge \mathbf{XX}\,(\alpha \wedge \varphi)$ imply $\alpha\,\mathring{\mathbf{W}}\,\varphi$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\alpha \wedge \gamma'))$ such that $\gamma' \in \text{LTLQ}^3 \cup \text{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ and $\alpha \notin \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\neg\alpha \vee \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha\,\mathring{\mathbf{W}}\,\varphi$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\neg\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg(\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

Induction step:

—Let $\gamma = \alpha \wedge \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \vee \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p]$, $\pi^{4n} \models \gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \wedge \neg\gamma'[p \wedge q])$ for all $n \in \mathbb{N}$. Thus, since φ implies $\alpha \vee \varphi$ and $\mathbf{G}\,(\neg\alpha \wedge \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{X}\,\gamma'$. By induction hypothesis, we obtain a path σ for γ'. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi^{4n} \models \mathbf{X}\,\gamma'[p]$, $\pi^{4n} \models \mathbf{X}\,\gamma'[q]$, and $\pi \models \mathbf{XG}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{XG}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{X}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \mathring{\mathbf{U}} \alpha = \gamma' \mathbf{U} (\gamma' \wedge \alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{G} \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi \mathring{\mathbf{U}} \alpha$ and $\mathbf{G} \neg\varphi$ implies $\mathbf{G} \neg(\varphi \mathring{\mathbf{U}} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \mathring{\mathbf{W}} \alpha = (\mathbf{G} \gamma') \vee (\gamma' \mathbf{U} (\gamma' \wedge \alpha))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \gamma'[p] \wedge \alpha$, $\pi^{4n} \models \gamma'[q] \wedge \alpha$, and $\pi \models \mathbf{G} \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\varphi \wedge \alpha$ implies $\varphi \mathring{\mathbf{W}} \alpha$ and $\mathbf{G} \neg\varphi$ implies $\mathbf{G} \neg(\varphi \mathring{\mathbf{W}} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathring{\mathbf{W}} \gamma' = (\mathbf{G} \alpha) \vee (\alpha \mathbf{U} (\alpha \wedge \gamma'))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$ and $\alpha \notin \ell(\pi^{4n+1})$ for all $n \in \mathbb{N}$. So we have $\pi^{4n} \models \alpha \wedge \gamma'[p]$, $\pi^{4n} \models \alpha \wedge \gamma'[q]$, and $\pi \models \mathbf{GF} \neg\alpha \wedge \mathbf{G} \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\alpha \wedge \varphi$ implies $\alpha \mathring{\mathbf{W}} \varphi$ and $\mathbf{GF} \neg\alpha \wedge \mathbf{G} \neg\varphi$ implies $\mathbf{G} \neg(\alpha \mathring{\mathbf{W}} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. □

Lemma 4.22. *Let $\gamma = \mathbf{G} \gamma''$ be a simple LTL query where $\gamma'' \in LTLQ^6$. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma \models \mathbf{G} \gamma'[p] \wedge \mathbf{G} \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Suppose further that for every $LTLQ^5$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi \models \mathbf{G} \gamma[p] \wedge \mathbf{G} \gamma[q]$ and $\pi \models \mathbf{G} \neg\gamma[p \wedge q]$.*

PROOF. Structural induction on γ.

Induction start:

—Let $\gamma = \mathbf{G} (\alpha \wedge \gamma')$ such that $\gamma' \in \text{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G} (\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G} (\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{GF} \neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G} (\alpha \wedge \varphi)$ is equivalent to $\mathbf{GG} (\alpha \wedge \varphi)$ and $\mathbf{GF} \neg\varphi$ implies $\mathbf{GF} \neg(\alpha \wedge \varphi)$ which is equivalent to $\mathbf{G} \neg\mathbf{G} (\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G} (\alpha \vee \gamma')$ such that $\gamma' \in \text{LTLQ}^5$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G} (\alpha \vee \gamma'[p])$, $\pi \models \mathbf{G} (\alpha \vee \gamma'[q])$, and $\pi \models \mathbf{GF} (\neg\alpha \wedge \neg\gamma'[p \wedge q])$. Thus, since $\mathbf{G} (\alpha \vee \varphi)$ is equivalent to $\mathbf{GG} (\alpha \vee \varphi)$ and $\mathbf{GF} (\neg\alpha \wedge \neg\varphi)$ is equivalent to $\mathbf{G} \neg\mathbf{G} (\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

Induction step:

—Let $\gamma = \mathbf{G} (\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for $\mathbf{G} \gamma'$. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G} (\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G} (\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{G} \neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G} (\alpha \wedge \varphi)$ is equivalent to $\mathbf{GG} (\alpha \wedge \varphi)$ and $\mathbf{G} \neg\varphi$ implies $\mathbf{GF} \neg(\alpha \wedge \varphi)$ which is equivalent to $\mathbf{G} \neg\mathbf{G} (\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}(\alpha \vee \gamma')$. By induction hypothesis, we obtain a path π for $\mathbf{G}\,\gamma'$. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}(\neg\alpha \wedge \neg\gamma'[p \wedge q])$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{GG}(\alpha \vee \varphi)$ and $\mathbf{G}(\neg\alpha \wedge \neg\varphi)$ implies $\mathbf{GF}(\neg\alpha \wedge \neg\varphi)$ which is equivalent to $\mathbf{G}\,\neg\mathbf{G}(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{GX}\,\gamma'$. By induction hypothesis, we obtain a path σ for $\mathbf{G}\,\gamma'$. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi \models \mathbf{XG}\,\gamma'[p]$, $\pi \models \mathbf{XG}\,\gamma'[q]$, and $\pi \models \mathbf{XG}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{XG}\,\varphi$ is equivalent to $\mathbf{GGX}\,\varphi$ and $\mathbf{XG}\,\neg\varphi$ implies $\mathbf{XGF}\,\neg\varphi$ which is equivalent to $\mathbf{G}\,\neg\mathbf{GX}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}(\gamma'\,\mathring{\mathbf{U}}\,\alpha) = \mathbf{G}(\gamma'\,\mathbf{U}(\gamma' \wedge \alpha))$. By induction hypothesis, we obtain a path π for $\mathbf{G}\,\gamma'$. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}(\gamma'[p] \wedge \alpha)$, $\pi \models \mathbf{G}(\gamma'[q] \wedge \alpha)$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}(\varphi \wedge \alpha)$ implies $\mathbf{GG}(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{GF}\,\neg(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ which is equivalent to $\mathbf{G}\,\neg\mathbf{G}(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}(\gamma'\,\mathring{\mathbf{W}}\,\alpha) = \mathbf{G}((\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}(\gamma' \wedge \alpha)))$. By induction hypothesis, we obtain a path π for $\mathbf{G}\,\gamma'$. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}(\gamma'[p] \wedge \alpha)$, $\pi \models \mathbf{G}(\gamma'[q] \wedge \alpha)$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}(\varphi \wedge \alpha)$ implies $\mathbf{GG}(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{GF}\,\neg(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ which is equivalent to $\mathbf{G}\,\neg\mathbf{G}(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. □

Lemma 4.23. *Let $\gamma \in LTLQ^3$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^4$ subquery γ' there exists a path σ such that $\sigma \models \mathbf{G}\,\gamma'[p] \wedge \mathbf{G}\,\gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Suppose further that for every $LTLQ^5$ and $LTLQ^6$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ for all $n \in \mathbb{N}$ and $\sigma \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Then, there exists a path π such that $\pi \models \mathbf{G}\,\gamma[p] \wedge \mathbf{G}\,\gamma[q]$ and $\pi \models \mathbf{G}\,\neg\gamma[p \wedge q]$.*

PROOF. Structural induction on γ.

Induction start:

—Let $\gamma = \mathbf{F}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By Lemma 4.20 resp. assumption, we obtain a path π for γ'. So we have $\pi \models \mathbf{GF}\,\gamma'[p]$, $\pi \models \mathbf{GF}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{F}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^4$. By assumption, we obtain a path π for γ'. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ is equivalent to $\mathbf{GG}\,\varphi$ and $\mathbf{GF}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{G}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^6$. Then, we obtain the statement by Lemma 4.22.

—Let $\gamma = \gamma'\,\mathring{\mathbf{U}}\,\alpha = \gamma'\,\mathbf{U}(\gamma' \wedge \alpha)$ such that $\gamma' \in \mathrm{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^{4n})$

and $\alpha \notin \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p] \wedge \mathbf{GF}\,\alpha$, $\pi \models \mathbf{G}\,\gamma'[q] \wedge \mathbf{GF}\,\alpha$, and $\pi \models \mathbf{G}\,(\neg\gamma'[p \wedge q] \vee \neg\alpha)$. Thus, since $\mathbf{G}\,\varphi \wedge \mathbf{GF}\,\alpha$ implies $\mathbf{G}\,(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ and $\mathbf{G}\,(\neg\varphi \vee \neg\alpha)$ is equivalent $\mathbf{G}\,\neg(\varphi \wedge \alpha)$ which implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{U}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathbf{U}\,\gamma'$ such that $\gamma' \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By Lemma 4.20 resp. assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\alpha \wedge \mathbf{GF}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\alpha \wedge \mathbf{GF}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\alpha \wedge \mathbf{GF}\,\varphi$ implies $\mathbf{G}\,(\alpha\,\mathbf{U}\,\varphi)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathbf{U}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathring{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\alpha \wedge \gamma')$ such that $\gamma' \in \mathrm{LTLQ}^1 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By Lemma 4.20 resp. assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\alpha \wedge \mathbf{GF}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\alpha \wedge \mathbf{GF}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\alpha \wedge \mathbf{GF}\,\varphi$ implies $\mathbf{G}\,(\alpha\,\mathring{\mathbf{U}}\,\varphi)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathring{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma')$ such that $\gamma' \in \mathrm{LTLQ}^4 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[p]) \wedge \mathbf{GF}\,\neg\alpha$, $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[q]) \wedge \mathbf{GF}\,\neg\alpha$, and $\pi \models \mathbf{G}\,(\alpha \vee \neg\gamma'[p \wedge q])$. Thus, since $\mathbf{G}\,(\alpha \vee \varphi) \wedge \mathbf{GF}\,\neg\alpha$ implies $\mathbf{G}\,(\alpha\,\bar{\mathbf{U}}\,\varphi)$ and $\mathbf{G}\,(\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\neg\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathbf{W}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\alpha)$ such that $\gamma' \in \mathrm{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\gamma'[p \wedge q] \wedge \mathbf{G}\,\neg\alpha$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\varphi\,\mathbf{W}\,\alpha)$ and $\mathbf{GF}\,\neg\varphi \wedge \mathbf{G}\,\neg\alpha$ implies $\mathbf{G}\,\neg(\varphi\,\mathbf{W}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathbf{W}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\alpha)$ such that $\gamma' \in \mathrm{LTLQ}^6$. By Lemma 4.22, we obtain a path π for $\mathbf{G}\,\gamma'$. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{GG}\,\gamma'[p]$, $\pi \models \mathbf{GG}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\mathbf{G}\,\gamma'[p \wedge q] \wedge \mathbf{G}\,\neg\alpha$. Thus, since $\mathbf{GG}\,\varphi$ implies $\mathbf{G}\,(\varphi\,\mathbf{W}\,\alpha)$ and $\mathbf{G}\,\neg\mathbf{G}\,\varphi \wedge \mathbf{G}\,\neg\alpha$ implies $\mathbf{G}\,\neg(\varphi\,\mathbf{W}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha))$ such that $\gamma' \in \mathrm{LTLQ}^4$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{GF}\,\neg\gamma'[p \wedge q] \wedge \mathbf{G}\,\neg\alpha$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ and $\mathbf{GF}\,\neg\varphi \wedge \mathbf{G}\,\neg\alpha$ implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,\gamma')$ such that $\gamma' \in \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[p])$, $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[q])$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,(\alpha \vee \varphi)$ implies $\mathbf{G}\,(\alpha\,\mathbf{W}\,\varphi)$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathbf{W}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \, \bar{\mathbf{W}} \, \gamma' = (\mathbf{G}\,\alpha) \vee (\alpha \, \mathbf{U} \, (\neg\alpha \wedge \gamma'))$ such that $\gamma' \in \mathrm{LTLQ}^4 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[p])$, $\pi \models \mathbf{G}\,(\alpha \vee \gamma'[q])$, and $\pi \models \mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\alpha \vee \neg\gamma'[p \wedge q])$. Thus, since $\mathbf{G}\,(\alpha \vee \varphi)$ implies $\mathbf{G}\,(\alpha \, \bar{\mathbf{W}} \, \varphi)$ and $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,(\alpha \vee \neg\varphi)$ is equivalent to $\mathbf{GF}\,\neg\alpha \wedge \mathbf{G}\,\neg(\neg\alpha \wedge \varphi)$ which implies $\mathbf{G}\,\neg(\alpha \, \bar{\mathbf{W}} \, \varphi)$ for all formulas φ, we obtain the statement for γ on π.

Induction step:

—Let $\gamma = \alpha \wedge \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G}\,(\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha \wedge \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \vee \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,(\neg\alpha \wedge \neg\gamma'[p \wedge q])$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\alpha \vee \varphi)$ and $\mathbf{G}\,(\neg\alpha \wedge \neg\varphi)$ is equivalent to $\mathbf{G}\,\neg(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{X}\,\gamma'$. By induction hypothesis, we obtain a path σ for γ'. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi \models \mathbf{XG}\,\gamma'[p]$, $\pi \models \mathbf{XG}\,\gamma'[q]$, and $\pi \models \mathbf{XG}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{XG}\,\varphi$ is equivalent to $\mathbf{GX}\,\varphi$ and $\mathbf{XG}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{X}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{F}\,\gamma'$. By induction hypothesis, we obtain a path π for γ'. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{GF}\,\varphi$ and $\mathbf{G}\,\neg\varphi$ is equivalent to $\mathbf{G}\,\neg\mathbf{F}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{G}\,\gamma'$. By induction hypothesis, we obtain a path π for γ'. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ is equivalent to $\mathbf{GG}\,\varphi$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{GF}\,\neg\varphi$ which is equivalent to $\mathbf{G}\,\neg\mathbf{G}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \, \mathring{\mathbf{U}} \, \alpha = \gamma' \, \mathbf{U} \, (\gamma' \wedge \alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\gamma'[p] \wedge \alpha)$, $\pi \models \mathbf{G}\,(\gamma'[q] \wedge \alpha)$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,(\varphi \wedge \alpha)$ implies $\mathbf{G}\,(\varphi \, \mathring{\mathbf{U}} \, \alpha)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\varphi \, \mathring{\mathbf{U}} \, \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \, \mathbf{U} \, \gamma'$. By induction hypothesis, we obtain a path π for γ'. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\alpha \, \mathbf{U} \, \varphi)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha \, \mathbf{U} \, \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \, \mathring{\mathbf{U}} \, \gamma' = \alpha \, \mathbf{U} \, (\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \in \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G}\,(\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,(\alpha \wedge \varphi)$ implies $\mathbf{G}\,(\alpha \, \mathring{\mathbf{U}} \, \varphi)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha \, \mathring{\mathbf{U}} \, \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{U}}\,\gamma' = \alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\neg\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G}\,(\neg\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,(\neg\alpha \wedge \varphi)$ implies $\mathbf{G}\,(\alpha\,\bar{\mathbf{U}}\,\varphi)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{U}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathbf{W}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\alpha)$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q] \wedge \mathbf{G}\,\neg\alpha$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\varphi\,\mathbf{W}\,\alpha)$ and $\mathbf{G}\,\neg\varphi \wedge \mathbf{G}\,\neg\alpha$ implies $\mathbf{G}\,\neg(\varphi\,\mathbf{W}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathring{\mathbf{W}}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,(\gamma' \wedge \alpha))$. By induction hypothesis, we obtain a path π for γ'. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ and $\mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\varphi\,\mathring{\mathbf{W}}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\mathbf{W}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,\gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi \models \mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}\,(\alpha\,\mathbf{W}\,\varphi)$ and $\mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\mathbf{W}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha\,\bar{\mathbf{W}}\,\gamma' = (\mathbf{G}\,\alpha) \vee (\alpha\,\mathbf{U}\,(\neg\alpha \wedge \gamma'))$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^n)$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\neg\alpha \wedge \gamma'[p])$, $\pi \models \mathbf{G}\,(\neg\alpha \wedge \gamma'[q])$, and $\pi \models \mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\gamma'[p \wedge q]$. Thus, since $\mathbf{G}\,(\neg\alpha \wedge \varphi)$ implies $\mathbf{G}\,(\alpha\,\bar{\mathbf{W}}\,\varphi)$ and $\mathbf{G}\,\neg\alpha \wedge \mathbf{G}\,\neg\varphi$ implies $\mathbf{G}\,\neg(\alpha\,\bar{\mathbf{W}}\,\varphi)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. ☐

Lemma 4.24. *Let $\gamma \in LTLQ^4$ be simple. Further, let p and q be atomic propositions not occurring in γ. Suppose that for every $LTLQ^3$, $LTLQ^5$, and $LTLQ^6$ subquery γ' there exists a path σ such that $\sigma^{4n} \models \gamma'[p] \wedge \gamma'[q]$ and $\sigma^{4n} \not\models \gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Then, there exists a path π such that $\pi \models \mathbf{G}\,\gamma[p] \wedge \mathbf{G}\,\gamma[q]$ and $\pi^{4n} \not\models \gamma[p \wedge q]$ for all $n \in \mathbb{N}$.*

PROOF. Structural induction on γ.

Induction start:

—Let $\gamma = \gamma'\,\mathbf{U}\,\alpha$ where $\gamma' \in \mathrm{LTLQ}^3 \cup \mathrm{LTLQ}^5 \cup \mathrm{LTLQ}^6$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\gamma'[p] \vee \alpha) \wedge \mathbf{GF}\,\alpha$, $\pi \models \mathbf{G}\,(\gamma'[q] \vee \alpha) \wedge \mathbf{GF}\,\alpha$, and $\pi^{4n} \models \neg\gamma'[p \wedge q] \wedge \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,(\varphi \vee \alpha) \wedge \mathbf{GF}\,\alpha$ implies $\mathbf{G}\,(\varphi\,\mathbf{U}\,\alpha)$ and $\neg\varphi \wedge \neg\alpha$ implies $\neg(\varphi\,\mathbf{U}\,\alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma'\,\mathbf{W}\,\alpha = (\mathbf{G}\,\gamma') \vee (\gamma'\,\mathbf{U}\,\alpha)$ where $\gamma' \in \mathrm{LTLQ}^5$. By assumption, we obtain a path π for γ'. Since $\alpha \notin \mathrm{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{[4n+1,4n+3]})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,(\gamma'[p] \vee \alpha)$, $\pi \models \mathbf{G}\,(\gamma'[q] \vee \alpha)$, and $\pi^{4n} \models \neg\gamma'[p \wedge q] \wedge \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,(\varphi \vee \alpha)$

implies $\mathbf{G}(\varphi \mathbf{W} \alpha)$ and $\neg\varphi \wedge \neg\alpha$ implies $\neg(\varphi \mathbf{W} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

Induction step:

—Let $\gamma = \alpha \vee \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi^{4n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}(\alpha \vee \varphi)$ and $\neg\alpha \wedge \neg\varphi$ is equivalent to $\neg(\alpha \vee \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \mathbf{X}\,\gamma'$. By induction hypothesis, we obtain a path σ for γ'. Let $\pi = s \circ \sigma$, where s is any state. So we have $\pi \models \mathbf{XG}\,\gamma'[p]$, $\pi \models \mathbf{XG}\,\gamma'[q]$, and $\pi^{4n} \models \mathbf{X}\,\neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{XG}\,\varphi$ is equivalent to $\mathbf{GX}\,\varphi$ and $\mathbf{X}\,\neg\varphi$ is equivalent to $\neg\mathbf{X}\,\varphi$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \gamma' \mathbf{U} \alpha$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ and $\alpha \in \ell(\pi^{4n+1})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p] \wedge \mathbf{GF}\,\alpha$, $\pi \models \mathbf{G}\,\gamma'[q] \wedge \mathbf{GF}\,\alpha$, and $\pi^{4n} \models \neg\gamma'[p \wedge q] \wedge \neg\alpha$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\varphi \wedge \mathbf{GF}\,\alpha$ implies $\mathbf{G}(\varphi \mathbf{U} \alpha)$ and $\neg\varphi \wedge \neg\alpha$ implies $\neg(\varphi \mathbf{U} \alpha)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathbf{U} \gamma'$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi^{4n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}(\alpha \mathbf{U} \varphi)$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha \mathbf{U} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

—Let $\gamma = \alpha \mathbf{W} \gamma' = (\mathbf{G}\,\alpha) \vee (\alpha \mathbf{U} \gamma')$. By induction hypothesis, we obtain a path π for γ'. Since $\alpha \notin \text{aprop}(\gamma')$, we can assume w.l.o.g. that $\alpha \notin \ell(\pi^{4n})$ for all $n \in \mathbb{N}$. So we have $\pi \models \mathbf{G}\,\gamma'[p]$, $\pi \models \mathbf{G}\,\gamma'[q]$, and $\pi^{4n} \models \neg\alpha \wedge \neg\gamma'[p \wedge q]$ for all $n \in \mathbb{N}$. Thus, since $\mathbf{G}\,\varphi$ implies $\mathbf{G}(\alpha \mathbf{W} \varphi)$ and $\neg\alpha \wedge \neg\varphi$ implies $\neg(\alpha \mathbf{W} \varphi)$ for all formulas φ, we obtain the statement for γ on π.

This concludes the proof. □

REFERENCES

BRUNS, G. AND GODEFROID, P. 2001. Temporal logic query checking. In *Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science (LICS)*. IEEE Computer Society, 409–417.

BUCCAFURRI, F., EITER, T., GOTTLOB, G., AND LEONE, N. 2001. On ACTL formulas having linear counterexamples. *Journal of Computer and System Sciences 62,* 3, 463–515.

CHAN, W. 2000. Temporal-logic queries. In *Proceedings of the 12th International Conference on Computer Aided Verification (CAV)*. Lecture Notes in Computer Science, vol. 1855. Springer-Verlag, 450–463.

CHECHIK, M. AND GURFINKEL, A. 2003. TLQSolver: A temporal logic query checker. In *Proceedings of the 15th International Conference on Computer Aided Verification (CAV)*. Lecture Notes in Computer Science, vol. 2725. Springer-Verlag, 210–214.

CLARKE, E. M., GRUMBERG, O., JHA, S., LU, Y., AND VEITH, H. 2000. Counterexample-guided abstraction refinement. In *Proceedings of the 12th International Conference on Computer Aided Verification (CAV)*. Lecture Notes in Computer Science, vol. 1855. Springer-Verlag, 154–169. Extended version available as [Clarke et al. 2003].

Clarke, E. M., Grumberg, O., Jha, S., Lu, Y., and Veith, H. 2003. Counterexample-guided abstraction refinement for symbolic model checking. *Journal of the ACM 50,* 5, 752–794.

Clarke, E. M., Grumberg, O., and Long, D. E. 1994. Model checking and abstraction. *ACM Transactions on Programming Languages and Systems (TOPLAS) 16,* 5, 1512–1542.

Clarke, E. M., Talupur, M., and Veith, H. 2006. Environment abstraction for parameterized verification. In *Proceedings of the 7th International Conference on Verification, Model Checking, and Abstract Interpretation (VMCAI).* Lecture Notes in Computer Science, vol. 3855. Springer-Verlag, 126–141.

Clarke, E. M., Talupur, M., and Veith, H. 2008. Proving Ptolemy right: The environment abstraction framework for model checking concurrent systems. In *Proceedings of the 14th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS).* Lecture Notes in Computer Science, vol. 4963. Springer-Verlag, 33–47.

Emerson, E. A. 1990. Temporal and modal logic. In *Handbook of Theoretical Computer Science*, J. van Leeuwen, Ed. Vol. B: Formal Methods and Semantics. Elsevier Science, Chapter 16, 995–1067.

Gurfinkel, A., Chechik, M., and Devereux, B. 2003. Temporal logic query checking: A tool for model exploration. *IEEE Transactions on Software Engineering (TSE) 29,* 10, 898–914.

Gurfinkel, A., Devereux, B., and Chechik, M. 2002. Model exploration with temporal logic query checking. In *Proceedings of the 10th ACM Symposium on Foundations of Software Engineering (FSE).* ACM, 139–148.

Hornus, S. and Schnoebelen, P. 2002. On solving temporal logic queries. In *Proceedings of the 9th International Conference on Algebraic Methodology and Software Technology (AMAST).* Lecture Notes in Computer Science, vol. 2422. Springer-Verlag, 163–177.

Maidl, M. 2000. The common fragment of CTL and LTL. In *Proceedings of the 41st Annual Symposium on Foundations of Computer Science (FOCS).* IEEE Computer Society, 643–652.

Samer, M. 2004. Reasoning about specifications in model checking. Ph.D. thesis, Vienna University of Technology.

Samer, M. and Veith, H. 2003. Validity of CTL queries revisited. In *Proceedings of the 12th Annual Conference of the European Association for Computer Science Logic (CSL).* Lecture Notes in Computer Science, vol. 2803. Springer-Verlag, 470–483.

Samer, M. and Veith, H. 2004a. Parameterized vacuity. In *Proceedings of the 5th International Conference on Formal Methods in Computer-Aided Design (FMCAD).* Lecture Notes in Computer Science, vol. 3312. Springer-Verlag, 322–336.

Samer, M. and Veith, H. 2004b. A syntactic characterization of distributive LTL queries. In *Proceedings of the 31st International Colloquium on Automata, Languages and Programming (ICALP).* Lecture Notes in Computer Science, vol. 3142. Springer-Verlag, 1099–1110.

Samer, M. and Veith, H. 2005. Deterministic CTL query solving. In *Proceedings of the 12th International Symposium on Temporal Representation and Reasoning (TIME).* IEEE Computer Society, 156–165.

Samer, M. and Veith, H. 2007. On the notion of vacuous truth. In *Proceedings of the 14th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR).* Lecture Notes in Computer Science, vol. 4790. Springer-Verlag, 2–14.

Sistla, A. P. and Clarke, E. M. 1985. The complexity of propositional linear temporal logics. *Journal of the ACM 32,* 3, 733–749.

Received September 2006; accepted June 2009